CHAPTER 5

Exponential and logarithmic functions

Objectives

- To graph exponential and logarithmic functions.
- To graph transformations of the graphs of exponential and logarithmic functions.
- To introduce Euler's number.
- To revise the index and logarithm laws.
- To solve exponential and logarithmic equations.
- To find rules for the graphs of exponential and logarithmic functions.
- To find inverses of exponential and logarithmic functions.
- To apply exponential functions to physical occurrences of exponential growth and decay.

5.1 Exponential functions

The function, \( f(x) = a^x \), where \( a \in R^+\setminus\{1\} \), is called an exponential function (or index function).

The graph of \( f(x) = a^x \) is shown.

The features of the graph of the exponential function with rule \( f(x) = a^x \) are:

- \( f(-1) = \frac{1}{a} \)
- \( f(0) = 1 \)
- \( f(1) = a \)
- The x-axis is a horizontal asymptote.
  - As \( x \to -\infty \), \( f(x) \to 0^+ \).
- The maximal domain is \( R \).
- The range of the function is \( R^+ \).
- An exponential function is a one-to-one function.
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Graphing transformations of the graph of $f(x) = a^x$

Translations
If the transformation, a translation with mapping $(x, y) \mapsto (x + h, y + k)$, is applied to the graph of $y = a^x$, the image has equation $y = a^{x-h} + k$. The horizontal asymptote has equation $y = k$. The images of the points with coordinates $\left(-1, \frac{1}{a}\right)$, $(0, 1)$ and $(1, a)$ are $(1 + h, a + k)$ respectively. The range of the image is $(k, \infty)$.

Example 1

Sketch the graph, and state the range, of $y = 2^{x-1} + 2$

Solution
A translation of 1 unit in the positive direction of the $x$-axis and 2 units in the positive direction of the $y$-axis is applied to the graph of $y = 2^x$.

The equation of the asymptote is $y = 2$

The mapping is $(x, y) \mapsto (x + 1, y + 2)$

$\left(-1, \frac{1}{2}\right) \rightarrow \left(0, \frac{5}{2}\right)$

$(0, 1) \rightarrow (1, 3)$

$(1, 2) \rightarrow (2, 4)$

The range of the function is $(2, \infty)$.

Reflections
If the transformation, a reflection in the $x$-axis determined by the mapping $(x, y) \mapsto (x, -y)$, is applied to the graph of $y = a^x$, the image has equation $y = -a^x$. The horizontal asymptote has equation $y = 0$. The images of the points with coordinates $\left(-1, \frac{1}{a}\right)$, $(0, 1)$ and $(1, a)$ are $\left(-1, -\frac{1}{a}\right)$, $(0, -1)$ and $(1, -a)$ respectively. The range of the image is $(-\infty, 0)$.

Example 2

Sketch the graph of $y = -3^x$
Solution

A reflection in the $x$-axis is applied to the graph of $y = 3^x$

The mapping is $(x, y) \rightarrow (x, -y)$

$$
\begin{align*}
(-1, \frac{1}{3}) & \rightarrow (-1, -\frac{1}{3}) \\
(0, 1) & \rightarrow (0, -1) \\
(1, 3) & \rightarrow (1, -3)
\end{align*}
$$

If the transformation, a reflection in the $y$-axis determined by the mapping $(x, y) \rightarrow (-x, y)$, is applied to the graph of $y = a^x$, the image has equation $y = a^{-x}$ (or $y = \frac{1}{a^x}$ or $y = \left(\frac{1}{a}\right)^x$).

The horizontal asymptote has equation $y = 0$. The images of the points with coordinates $(-1, \frac{1}{a}), (0, 1)$ and $(1, a)$ are $(1, \frac{1}{a}), (0, 1)$ and $(-1, a)$ respectively. The range of the image is $(0, \infty)$.

Example 3

Sketch the graph of $y = 6^{-x}$

Solution

A reflection in the $y$-axis is applied to the graph of $y = 6^x$

The mapping is $(x, y) \rightarrow (-x, y)$

$$
\begin{align*}
(-1, \frac{1}{6}) & \rightarrow (1, \frac{1}{6}) \\
(0, 1) & \rightarrow (0, 1) \\
(1, 6) & \rightarrow (-1, 6)
\end{align*}
$$

Dilations

If the transformation, a dilation of factor $k$ $(k > 0)$ from the $x$-axis determined by the mapping $(x, y) \rightarrow (x, ky)$, is applied to the graph of $y = a^x$, the image has equation $y = ka^x$.

The horizontal asymptote has equation $y = 0$. The images of the points with coordinates $(-1, \frac{1}{a}), (0, 1)$ and $(1, a)$ are $(-1, \frac{k}{a}), (0, k)$ and $(1, ka)$ respectively. The range of the image is $(0, \infty)$.

Example 4

Sketch the graph of each of the following:

a $y = 3(5)^x$

b $y = (0.2)(8)^x$
Solution

a A dilation of factor 3 from the x-axis is applied to the graph of $y = 5^x$
The mapping is $(x, y) \rightarrow (x, 3y)$

- $(-1, \frac{1}{5}) \rightarrow (-1, \frac{3}{5})$
- $(0, 1) \rightarrow (0, 3)$
- $(1, 5) \rightarrow (1, 15)$

b A dilation of factor 0.2 (or $\frac{1}{5}$) from the x-axis is applied to the graph of $y = 8^x$
The mapping is $(x, y) \rightarrow (x, \frac{1}{5}y)$

- $(-1, \frac{1}{8}) \rightarrow (-1, \frac{1}{40})$
- $(0, 1) \rightarrow \left(0, \frac{1}{5}\right)$
- $(1, 8) \rightarrow \left(1, \frac{8}{5}\right)$

If the transformation, a dilation of factor $k$ ($k > 0$) from the y-axis determined by the mapping $(x, y) \rightarrow (kx, y)$, is applied to the graph of $y = a^x$, the image has equation $y = a^x$. The horizontal asymptote has equation $y = 0$. The images of the points with coordinates $\left(-1, \frac{1}{a}\right)$, $(0, 1)$ and $(1, a)$ are $\left(-k, \frac{1}{a}\right)$, $(0, 1)$ and $(k, a)$ respectively. The range of the image is $(0, \infty)$.

Example 5

Sketch the graph of each of the following:

a $y = 9^x$  
b $y = 2^3x$

Solution

a A dilation of factor 2 from the y-axis is applied to the graph of $y = 9^x$
The mapping is $(x, y) \rightarrow (2x, y)$

- $(-1, \frac{1}{9}) \rightarrow (-2, \frac{1}{9})$
- $(0, 1) \rightarrow (0, 1)$
- $(1, 9) \rightarrow (2, 9)$
A dilation of factor $\frac{1}{3}$ from the $y$-axis is applied to the graph of $y = 2^x$.

The mapping is $(x, y) \rightarrow \left( \frac{1}{3}x, y \right)$

$(-1, \frac{1}{2}) \rightarrow \left( -\frac{1}{3}, \frac{1}{2} \right)$

$(0, 1) \rightarrow (0, 1)$

$(1, 2) \rightarrow \left( \frac{1}{3}, 2 \right)$

The range of the function is $(3, \infty)$.

The transformations, a reflection in the $y$-axis and a translation of 3 units in the positive direction of the $y$-axis, are applied to the graph of $y = 4^x$.

The equation of the asymptote is $y = -1$

The mapping is $(x, y) \rightarrow \left( \frac{1}{3}x, y - 1 \right)$

$(-1, \frac{1}{4}) \rightarrow \left( -\frac{1}{3}, -\frac{3}{4} \right)$

$(0, 1) \rightarrow (0, 0)$

$(1, 4) \rightarrow \left( \frac{1}{3}, 3 \right)$

The range of the function is $(-1, \infty)$.
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The transformations, a reflection in the x-axis followed by a translation of 1 unit in the positive direction of the x-axis and 2 units in the negative direction of the y-axis, are applied to the graph of \( y = 10^x \).

The equation of the asymptote is \( y = -2 \).

The mapping is \((x, y) \rightarrow (x + 1, -y - 2)\).

\[
\begin{align*}
(-1, \frac{1}{10}) & \rightarrow (0, -\frac{21}{10}) \\
(0, 1) & \rightarrow (1, -3) \\
(1, 10) & \rightarrow (2, -12)
\end{align*}
\]

The range of the function is \(( -\infty, -2 )\).

### Exercise 5A

1. For each of the following, use the one set of axes to sketch the graphs (and label asymptotes) of:
   - **a** \( y = 2^x \) and \( y = 3^x \)
   - **b** \( y = 2^{-x} \) and \( y = 3^{-x} \)
   - **c** \( y = 5^x \) and \( y = -5^x \)
   - **d** \( y = (1.5)^x \) and \( y = -(1.5)^x \)

2. Sketch the graph of each of the following (labelling asymptotes), and state the range of each:
   - **a** \( y = 2^{x+1} - 2 \)
   - **b** \( y = \left(\frac{1}{2}\right)^{x-2} \)
   - **c** \( y = \left(\frac{1}{2}\right)^x - 1 \)
   - **d** \( y = \left(\frac{1}{2}\right)^{-x} + 1 \)
   - **e** \( y = 2^{x-2} + 2 \)
   - **f** \( y = \left(\frac{1}{2}\right)^{-x-2} + 2 \)

3. Sketch the graph of each of the following (labelling asymptotes), and state the range of each:
   - **a** \( y = 3^x \)
   - **b** \( y = 3^x + 1 \)
   - **c** \( y = 1 - 3^x \)
   - **d** \( y = \left(\frac{1}{3}\right)^x \)
   - **e** \( y = 3^{-x} + 2 \)
   - **f** \( y = \left(\frac{1}{3}\right)^{-x} - 1 \)

4. For \( f(x) = 2^x \), sketch the graph of each of the following, labelling asymptotes where appropriate:
   - **a** \( y = f(x + 1) \)
   - **b** \( y = f(x) + 1 \)
   - **c** \( y = f(-x) + 2 \)
   - **d** \( y = -f(x) - 1 \)
   - **e** \( y = f(3x) \)
   - **f** \( y = f\left(\frac{x}{2}\right) \)
   - **g** \( y = 2f(x - 1) + 1 \)
   - **h** \( y = f(x - 2) \)

5. Sketch the graph of each of the following (labelling asymptotes), and state the range of each:
   - **a** \( y = 10^x - 1 \)
   - **b** \( y = 10^{\frac{x}{10}} + 1 \)
   - **c** \( y = 2 \times 10^x - 20 \)
   - **d** \( y = 1 - 10^{-x} \)
   - **e** \( y = 10^{x+1} + 3 \)
   - **f** \( y = 2 \times 10^{\frac{x}{10}} + 4 \)
6. A bank offers cash loans at .04% interest per day compounded daily. A loan of $10 000 is taken and the interest payable at the end of $x$ days is given by $C_1 = 10 000 \left((1.0004)^x - 1\right)$.

   a. Plot the graph of $C_1$ against $x$.
   b. Find the interest at the end of:
      i. 100 days
      ii. 300 days
   c. After how many days is the interest payable $1000$?
   d. A loan company offers $10 000 with a charge of $4.25 a day being made. The amount charged after $x$ days is given by $C_2 = 4.25x$
      i. Plot the graph of $C_2$ against $x$ (using the same window as in a).
      ii. Find the smallest value of $x$ for which $C_2 < C_1$.

7. If you invest $100 at 2% per day, compounding daily, the amount of money you would have after $x$ days is given by $y = 100(1.02)^x$ dollars. For how many days would you have to invest to double your money?

8. a. i. Graph $y = 2^x$, $y = 3^x$ and $y = 5^x$ on the same set of axes.
      ii. For what values of $x$ is $2^x > 3^x > 5^x$?
      iii. For what values of $x$ is $2^x < 3^x < 5^x$?
      iv. For what values of $x$ is $2^x = 3^x = 5^x$?
   b. Repeat part a for $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$ and $y = \left(\frac{1}{5}\right)^x$.
   c. Use your answers to parts a and b to sketch the graph of $y = a^x$ for:
      i. $a > 1$
      ii. $a = 1$
      iii. $0 < a < 1$

5.2 The exponential function, $f(x) = e^x$

In the previous section the family of exponential functions $f(x) = a^x$, $a \in \mathbb{R}^+ \setminus \{1\}$, was explored. One member of this family is of such importance in mathematics that it is known as the exponential function. This function has the rule $f(x) = e^x$, where $e$ is Euler’s number, named after an eighteenth century Swiss mathematician.

Euler’s number is defined as:

$$ e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n $$

To see what the value of $e$ might be, we could try large values of $n$ and a calculator to evaluate $\left(1 + \frac{1}{n}\right)^n$.

Try $n = 100$ then $\left(1 + \frac{1}{100}\right)^{100} = (1.01)^{100} = 2.7048\ldots$ $n = 1000$ $(1.001)^{1000} = 2.7169\ldots$ $n = 10000$ $(1.0001)^{10000} = 2.7181\ldots$ $n = 100000$ $(1.0001)^{100000} = 2.71826\ldots$ $n = 1000000$ $(1.00001)^{1000000} = 2.71828\ldots$

As $n$ is taken larger and larger it can be seen that $\left(1 + \frac{1}{n}\right)^n$ approaches a limiting value ($\approx 2.71828$). Like $\pi$, $e$ is irrational: $e = 2.7182818284590452353\ldots$
Investigation into the production of glass marbles

A method of producing high quality glass marbles has been proposed. A rack holding small silica ‘cones’ threaded on a wire will circulate around the track as shown in the diagram. When the rack enters the spray unit it will be subjected to a fine spray of a liquid glass substance. It takes 1 minute to produce a marble.

A marble produced by a single passage around the unit will take 1 minute and the volume will be increased by 100%, i.e. doubled. However, such a large increase in volume, at this slow speed, will tend to produce misshapen marbles. This suggests that the rack should be speeded up. We shall investigate what happens to the volume of the marble as the rack is speeded up and try to answer the question, ‘Is there a maximum volume reached if the rack speeds up indefinitely?’

Let $V =$ volume of the marble at time $t$.

Also let the original marble volume equal $V_0$.

For 1 passage per minute $V = 2 \times V_0$.

Now assume that if the rack is speeded up to do 2 passages/minute then the growth in volume is 50% for each passage; that is:

$$V = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \quad V = \left( 1 + \frac{1}{2} \right)^2 \quad V = 2.25V_0$$

and similarly:

for 4 passages, $V = \left( 1 + \frac{1}{4} \right)^4 V_0 = 2.441 \ldots V_0$

for 8 passages, $V = \left( 1 + \frac{1}{8} \right)^8 V_0 = 2.565 \ldots V_0$

for 16 passages, $V = \left( 1 + \frac{1}{16} \right)^{16} V_0 = 2.637 \ldots V_0$

for 64 passages, $V = \left( 1 + \frac{1}{64} \right)^{64} V_0 = 2.697 \ldots V_0$

for $n$ passages, $V = \left( 1 + \frac{1}{n} \right)^n V_0$

As the rack speeds up, $n$ is taken larger and larger, and it can be seen that $\left( 1 + \frac{1}{n} \right)^n$ approaches a limiting value, i.e.

$$V = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n V_0$$

$$= eV_0$$

So the maximum volume of the marble if the rack speeds up indefinitely is $eV_0$. 

Graphing $y = e^x$

The graph of $y = e^x$ is as shown.

The graphs of $y = 2^x$ and $y = 3^x$ are shown on the same set of axes:

Exercise 5B

1 Sketch the graph of each of the following and state the range:
   a $f(x) = e^x + 1$
   b $f(x) = 1 - e^x$
   c $f(x) = 1 - e^{-x}$
   d $f(x) = e^{-2x}$
   e $f(x) = e^{x-1} - 2$
   f $f(x) = 2e^x$
   g $h(x) = 2(1 + e^x)$
   h $h(x) = 2(1 - e^{-x})$
   i $g(x) = 2e^{-x} + 1$
   j $h(x) = 2e^{x-1}$
   k $f(x) = 3e^{x+1} - 2$
   l $h(x) = 2 - 3e^x$

2 Solve each of the following equations using a calculator. Give answers correct to three decimal places.
   a $e^x = x + 2$
   b $e^{-x} = x + 2$
   c $x^2 = e^x$
   d $x^3 = e^x$

3 a Using a calculator plot the graph of $y = f(x)$ where $f(x) = e^x$
   b Using the same screen plot the graphs of:
      i $y = f(x - 2)$
      ii $y = f\left(\frac{x}{3}\right)$
      iii $y = f(-x)$

5.3 Exponential equations

In this section the one-to-one property of exponential functions is exploited to solve exponential equations. This property can be stated as:

$a^x = a^y$ implies $x = y$

Example 7

Find the value of $x$ for which:
   a $4^x = 256$
   b $3^{x-1} = 81$

Solution

   a $4^x = 256$
   b $3^{x-1} = 81$
   \[ 4^x = 4^4 \]
   \[ 3^{x-1} = 3^4 \]
   \[ \therefore x = 4 \]
   \[ \therefore x - 1 = 4 \]
   \[ x = 5 \]
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**Index laws**
The solution of equations may also require an application of one or more of the index laws and these are stated here:

- To multiply two numbers in exponent form with the same base, **add** the exponents:
  \[ a^m \times a^n = a^{m+n} \]

- To divide two numbers in exponent form with the same base, **subtract** the exponents:
  \[ a^m \div a^n = a^{m-n} \]

- To raise the power of \( a \) to another power, **multiply** the exponents:
  \[ (a^m)^n = a^{mn} \]

\[ a^0 = 1 \]

**Example 8**
Find the value of \( x \) for which:

\[ a \quad 5^{2x-4} = 25^{-x+2} \quad b \quad 9^x = 12 \times 3^x - 27 \]

**Solution**

\[ a \quad 5^{2x-4} = 25^{-x+2} \]
\[ = (5^2)^{-x+2} \]
\[ = 5^{-2x+4} \]
\[ \therefore \ 2x - 4 = -2x + 4 \]
\[ 4x = 8 \]
\[ x = 2 \]

\[ b \quad (3^x)^2 = 12 \times 3^x - 27 \]
Let \( y = 3^x \).
\[ \therefore \ y^2 = 12y - 27 \]
\[ y^2 - 12y + 27 = 0 \]
\[ (y - 3)(y - 9) = 0 \]
\[ \therefore \ y = 3 \text{ or } y = 9 \]
\[ \therefore \ 3^x = 3 \text{ or } 3^x = 3^2 \]
\[ x = 1 \text{ or } x = 2 \]

**Exercise 5C**
1. Simplify the following expressions:
   
   \[ a \quad 3x^2y^3 \times 2x^4y^6 \]
   \[ b \quad \frac{12x^8}{4x^2} \]
   \[ c \quad 18x^2y^3 \div 3x^4y \]
   \[ d \quad (4x^4y^2)^2 + 2(x^2y)^4 \]
   \[ e \quad (4x^0)^2 \]
   \[ f \quad 15(x^5y^{-3})^4 \div 3(x^4y)^{-2} \]
   \[ g \quad \frac{3(2x^2y^3)^4}{2x^3y^2} \]
   \[ h \quad (8x^3y^3)^\frac{1}{3} \]
   \[ i \quad \frac{x^2 + y^2}{x^2 - y^2} \]

2. Solve for \( x \) in each of the following:

   \[ a \quad 3^x = 81 \]
   \[ b \quad 81^x = 9 \]
   \[ c \quad 4^x = 256 \]
   \[ d \quad 625^x = 5 \]
   \[ e \quad 32^x = 8 \]
   \[ f \quad 5^x = 125 \]
   \[ g \quad 16^x = 1024 \]
   \[ h \quad 2^{-x} = \frac{1}{64} \]
   \[ i \quad 5^{-x} = \frac{1}{625} \]

3. Solve for \( n \) in each of the following:

   \[ a \quad 5^{2n} \times 25^{2n-1} = 625 \]
   \[ b \quad 4^{2n-2} = 1 \]
   \[ c \quad 4^{2n-1} = \frac{1}{256} \]
Solve for \( x \):

\[
\begin{align*}
\text{a} & \quad 3^{2x} - 12(3^x) + 27 = 0 \\
\text{b} & \quad 3^{2x} - 2(3^x) - 3 = 0 \\
\text{c} & \quad 5^{2x} - 23(5^x) - 50 = 0 \\
\text{d} & \quad 5^{2x} - 10(5^x) + 25 = 0 \\
\text{e} & \quad 2^{2x} - 6(2^x) - 8 = 0 \\
\text{f} & \quad 8(3^x) - 6 = 2(32^x) \\
\text{g} & \quad 2^{2x} - 20(2^x) = -64 \\
\text{h} & \quad 4^{2x} - 5(4^x) = -4 \\
\text{i} & \quad 3(3^{2x}) = 28(3^x) - 9 \\
\text{j} & \quad 7(7^{2x}) = 8(7^x) - 1
\end{align*}
\]

### 5.4 Logarithmic functions

The exponential function \( f(x) = a^x \), where \( a \in R^+\setminus\{1\} \), is a one-to-one function. Therefore, there exists an inverse function (see Section 1.7). To find the rule of the inverse function, do the following.

Let \( x = a^y \)

Therefore \( y = \log_a x \)

Therefore \( f^{-1}(x) = \log_a x \)

The following definition was used to find the rule of the inverse:

\[
\log_a x = y \text{ if } a^y = x
\]

For example:

\[
\begin{align*}
\log_2 8 & = 3 \text{ is equivalent to the statement } 2^3 = 8 \\
\log_{10} 0.1 & = -1 \text{ is equivalent to the statement } 10^{-1} = 0.1
\end{align*}
\]

The graphs of \( y = e^x \) and its inverse function \( y = \log_e x \) are shown on the one set of axes.

The graphs of \( y = \log_2 x \), \( y = \log_e x \) and \( y = \log_3 x \) are shown on the one set of axes.

The features of the graph of the logarithmic function with rule \( f(x) = \log_a x \) are:

- \( f(1) = 0 \)
- \( f(a) = 1 \)
The $y$-axis is a vertical asymptote. As $x \to 0^+$, $f(x) \to -\infty$.

The maximal domain is $R^+$.

The range of the function is $R$.

A logarithmic function is a one-to-one function.

Note: The function with rule $f(x) = \log_e x$ is known as the natural logarithm function.

Logarithm laws

We use the index laws to establish rules for computations with logarithms.

Let $a^x = m$ and $a^y = n$, where $m$, $n$ and $a$ are positive real numbers.

\[
\therefore mn = a^x \times a^y = a^{x+y}
\]

\[
\log_a (mn) = x + y \quad \text{and since} \quad x = \log_a m \quad \text{and} \quad y = \log_a n \quad \text{it follows that:}
\]

\[
\log_a (mn) = \log_a m + \log_a n
\]

For example:

\[
\log_{10} 200 + \log_{10} 5 = \log_{10} (200 \times 5) = \log_{10} (1000) = 3
\]

\[
\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}
\]

\[
\therefore \log_a \left(\frac{m}{n}\right) = x - y
\]

and so \[
\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n
\]

For example:

\[
\log_2 32 - \log_2 8 = \log_2 \frac{32}{8} = \log_2 4 = 2
\]

If $m = 1$

\[
\log_a \left(\frac{1}{n}\right) = \log_a 1 - \log_a n = -\log_a n
\]

\[
\therefore \log_a \left(\frac{1}{n}\right) = -\log_a n
\]

\[
m^p = (a^x)^p = a^{xp}
\]

\[
\log_a (m^p) = xp \quad \text{and so} \quad \log_a (m^p) = p \log_a m
\]

For example:

\[
3 \log_2 5 = \log_2 (5^3) = \log_2 125
\]
Example 9

Without using a calculator, simplify the following:

\[2 \log_{10} 3 + \log_{10} 16 - 2 \log_{10} \left(\frac{6}{5}\right)\]

Solution

\[2 \log_{10} 3 + \log_{10} 16 - 2 \log_{10} \left(\frac{6}{5}\right) = \log_{10} 3^2 + \log_{10} 16 - \log_{10} \left(\frac{6}{5}\right)^2\]
\[= \log_{10} 9 + \log_{10} 16 - \log_{10} \frac{36}{25}\]
\[= \log_{10} \left(9 \times 16 \times \frac{25}{36}\right)\]
\[= 2\]

Logarithmic equations

Example 10

Solve each of the following equations for \(x\):

a  \(\log_2 x = 5\)  
b  \(\log_2 (2x - 1) = 4\)  
c  \(\log_e (3x + 1) = 0\)

Solution

a  \(\log_2 x = 5\)  
\[\therefore x = 2^5\]
\[\therefore x = 32\]

b  \(\log_2 (2x - 1) = 4\)  
\[\therefore 2x - 1 = 2^4\]
\[\therefore 2x = 17\]
\[\therefore x = \frac{17}{2}\]

c  \(\log_e (3x + 1) = 0\)  
\[\therefore 3x + 1 = e^0\]
\[\therefore 3x = 1 - 1\]
\[\therefore x = 0\]

Using the TI-Nspire

Use \texttt{Solve( )} from the \texttt{Algebra} menu (menu arrow) as shown.

\(\ln(x) = \log_e(x)\), the logarithm with base \(e\), is available on the keypad by pressing \(\texttt{ln}\). Logarithms with other bases are obtained by pressing the \texttt{log} key (menu arrow) and completing the template.
Using the Casio ClassPad

Enter and highlight
\[ \ln(x - 1) + \ln(x + 2) = \ln(6x - 8) \]
then tap Interactive–Equation/inequality–solve. Ensure the variable is set to \( x \).

Example 11

Solve each of the following equations for \( x \):

a \( \log_x 27 = \frac{3}{2} \)

b \( \log_e (x - 1) + \log_e (x + 2) = \log_e (6x - 8) \)

c \( \log_2 x - \log_2 (7 - 2x) = \log_2 6 \)

Solution

a \[ \log_x 27 = \frac{3}{2} \]

is equivalent to \( x^{\frac{3}{2}} = 27 \)

By inspection \( x = 9 \)

b \[ \log_e (x - 1) + \log_e (x + 2) = \log_e (6x - 8) \]

\[ \therefore \log_e (x - 1)(x + 2) = \log_e (6x - 8) \]

\[ \therefore x^2 + x - 2 = 6x - 8 \]

\[ \therefore x^2 - 5x + 6 = 0 \]

\[ \therefore (x - 3)(x - 2) = 0 \]

\[ \therefore x = 3 \text{ or } x = 2 \]

Note: The solutions must satisfy \( x - 1 > 0, x + 2 > 0 \) and \( 6x - 8 > 0 \), i.e. \( x > \frac{4}{3} \).

Therefore both of these solutions are allowable.

c \[ \log_2 x - \log_2 (7 - 2x) = \log_2 6 \]

\[ \therefore \log_2 \frac{x}{7 - 2x} = \log_2 6 \]

\[ \therefore \frac{x}{7 - 2x} = 6 \]

\[ \therefore x = 42 - 12x \]

\[ \therefore 13x = 42 \]

\[ \therefore x = \frac{42}{13} \]
Graphing transformations of the graph of \( f(x) = \log_a x \)

**Example 12**

Sketch the graph of \( y = 3 \log_e 2x \)

**Solution**

This is obtained from the graph of \( y = \log_e x \) by a dilation of factor 3 from the \( x \)-axis and a dilation of factor \( \frac{1}{2} \) from the \( y \)-axis.

The mapping is \((x, y) \rightarrow \left(\frac{1}{2}x, 3y\right)\)

\[
(1, 0) \rightarrow \left(\frac{1}{2}, 0\right)
\]

\[
(e, 1) \rightarrow \left(\frac{1}{2}e, 3\right)
\]

**Example 13**

Sketch the graph, and state the domain, of each of the following:

a) \( y = \log_2 (x - 5) + 1 \)  

b) \( y = -\log_3 (x + 4) \)

**Solution**

a) The graph of \( y = \log_2 (x - 5) + 1 \) is obtained from the graph of \( y = \log_2 x \) by a translation of 5 units in the positive direction of the \( x \)-axis and 1 unit in the positive direction of the \( y \)-axis.

The equation of the asymptote is \( x = 5 \)

The mapping is \((x, y) \rightarrow (x + 5, y + 1)\)

\[
(1, 0) \rightarrow (6, 1)
\]

\[
(2, 1) \rightarrow (7, 2)
\]

The domain of the function is \((5, \infty)\).

When \( y = 0 \),

\[
\log_2 (x - 5) + 1 = 0
\]

\[
\log_2 (x - 5) = -1
\]

\[
x - 5 = 2^{-1}
\]

\[
x = 5 \frac{1}{2}
\]
The graph of \( y = -\log_3 (x + 4) \) is obtained from the graph of \( y = \log_3 x \) by a translation of 4 units in the negative direction of the \( x \)-axis and a reflection in the \( x \)-axis.

The equation of the asymptote is \( x = -4 \)

The mapping is \((x, y) \rightarrow (x - 4, -y)\)

\((1, 0) \rightarrow (-3, 0)\)
\((3, 1) \rightarrow (-1, -1)\)

The domain of the function is \((-4, \infty)\).

When \( x = 0 \), \( y = -\log_3 (0 + 4) = -\log_3 4 \)

\[ \text{Example 14} \]

Sketch the graph of \( y = 2 \log_e (x + 5) - 3 \).

**Solution**

The graph of \( y = 2 \log_e (x + 5) - 3 \) is obtained from the graph of \( y = \log_e x \) by a dilation of factor 2 from the \( x \)-axis followed by a translation of 5 units in the negative direction of the \( x \)-axis and 3 units in the negative direction of the \( y \)-axis.

The equation of the asymptote is \( x = -5 \)

The mapping is \((x, y) \rightarrow (x - 5, 2y - 3)\)

\((1, 0) \rightarrow (-4, -3)\)
\((e, 1) \rightarrow (e - 5, -1)\)

The domain of the function is \((-5, \infty)\).

When \( x = 0 \), \( y = 2 \log_e (0 + 5) - 3 \)
\[ = 2 \log_e 5 - 3 \]

When \( y = 0 \),
\[ 2 \log_e (x + 5) - 3 = 0 \]
\[ \Rightarrow \log_e (x + 5) = \frac{3}{2} \]
and
\[ x + 5 = e^{\frac{3}{2}} \]
\[ \Rightarrow x = e^{\frac{3}{2}} - 5 \]
\[ \approx -0.518 \]
Exercise 5D

1 Evaluate each of the following:
   a \( \log_{10} 1000 \)
   b \( \frac{1}{16} \)
   c \( 0.001 \)
   d \( \log_2 64 \)
   e \( \log_{10} 1\,000\,000 \)
   f \( \frac{1}{128} \)

2 Express the following as the logarithm of a single term:
   a \( \log_e 2 + \log_e 3 \)
   b \( \log_e 32 - \log_e 8 \)
   c \( \log_e 10 + \log_e 100 + \log_e 1000 \)
   d \( \log_e \frac{1}{2} + \log_e 14 \)
   e \( \log_e \frac{1}{3} + \log_e \frac{1}{4} + \log_e \frac{1}{5} \)
   f \( \log_e uv + \log_e uv^2 + \log_e uv^3 \)
   g \( 2 \log_e x + 5 \log_e y \)
   h \( \log_e (x + y) + \log_e (x - y) - \log_e (x^2 - y^2) \)

3 Solve each of the following equations for \( x \):
   a \( \log_{10} x = 2 \)
   b \( 2 \log_2 x = 8 \)
   c \( \log_e (x - 5) = 0 \)
   d \( \log_2 x = 6 \)
   e \( 2 \log_e (x + 5) = 6 \)
   f \( \log_e (2x) = 0 \)
   g \( \log_e (2x + 3) = 0 \)
   h \( \log_{10} x = -3 \)
   i \( 2 \log_2 (x - 4) = 10 \)

4 Solve each of the following equations for \( x \):
   a \( \log_{10} x = \log_{10} 3 + \log_{10} 5 \)
   b \( \log_e x = \log_e 15 - \log_e 3 \)
   c \( \log_e x = \frac{2}{3} \log_e 8 \)
   d \( \log_e x + \log_e (2x - 1) = 0 \)
   e \( 2 \log_e x - \log_e (x - 1) = \log_e (x + 3) \)

5 Express each of the following as the logarithm of a single term:
   a \( \log_{10} 9 + \log_{10} 3 \)
   b \( \log_2 24 - \log_2 6 \)
   c \( \frac{1}{2} \log_{10} a - \frac{1}{2} \log_{10} b \)
   d \( 1 + \log_{10} a - \frac{1}{3} \log_{10} b \)
   e \( \frac{1}{2} \log_{10} 36 - \frac{1}{3} \log_{10} 27 - \frac{2}{3} \log_{10} 64 \)

6 Without using your calculator, evaluate each of the following:
   a \( \log_{10} 5 + \log_{10} 2 \)
   b \( \log_{10} 5 + 3 \log_{10} 2 - \log_{10} 4 \)
   c \( \log_2 \sqrt{2} + \log_2 1 + 2 \log_2 2 \)
   d \( 2 \log_{10} 5 + 2 \log_{10} 2 + 1 \)
   e \( 4 \log_{10} 2 - \log_{10} 16 \)

7 Simplify the following expressions:
   a \( \log_3 \left( \frac{1}{3^x} \right) \)
   b \( \log_2 x - 2 \log_2 y + \log_2 (xy^2) \)
   c \( \log_e (x^2 - y^2) - \log_e (x - y) - \log_e (x + y) \)

8 Solve each of the following equations for \( x \):
   a \( \log_e (x^2 - 2x + 8) = 2 \log_e x \)
   b \( \log_e (5x) - \log_e (3 - 2x) = 1 \)
Chapter 5 — Exponential and logarithmic functions

9 Solve each of the following equations for x:
   a \( \log_e (x) + \log_e (3x + 1) = 1 \)
   b \( 8e^{-x} - e^x = 2 \)

10 Solve for x:
   \( 2 \log_e (x) + \log_e 4 = \log_e (9x - 2) \)

11 Given that \( \log_a N = \frac{1}{2} (\log_a 24 - \log_a 0.375 - 6 \log_a 3) \), find the value of N.

12 Sketch the graphs of each of the following. Label the axes intercepts and asymptotes. State the maximal domain and range of each.
   a \( y = 2 \log_e (x - 3) \)
   b \( y = \log_e (x + 3) + 2 \)
   c \( y = 2 \log_e (x + 1) - 1 \)
   d \( y = 2 + \log_e (3x - 2) \)
   e \( y = -2 \log_e (x + 2) \)
   f \( y = -2 \log_e (x - 2) \)
   g \( y = 1 - \log_e (x + 1) \)
   h \( y = \log_e (2 - x) \)
   i \( y + 1 = \log_e (4 - 3x) \)

13 Sketch the graphs of each of the following. Label the axes intercepts and asymptotes. State the maximal domain of each.
   a \( y = \log_2 2x \)
   b \( y = \log_{10} (x - 5) \)
   c \( y = -\log_{10} x \)
   d \( y = \log_{10} (-x) \)
   e \( y = \log_{10} (5 - x) \)
   f \( y = 2 \log_2 2x + 2 \)
   g \( y = -2 \log_2 3x \)
   h \( y = \log_{10} (-x - 5) + 2 \)
   i \( y = 4 \log_2 (-3x) \)
   j \( y = 2 \log_2 (2 - x) - 6 \)

14 Solve each of the following equations using a calculator. Give answers correct to three decimal places.
   a \( -x + 2 = \log_e x \)
   b \( \frac{1}{3} \log_e (2x + 1) = -\frac{1}{2}x + 1 \)

15 a Using a calculator plot the graph of \( y = f(x) \) where \( f(x) = \log_e x \)
   b Using the same screen plot the graphs of:
      i \( y = f(-x) \)
      ii \( y = -f(x) \)
      iii \( y = f \left( \frac{x}{3} \right) \)
      iv \( y = f(3x) \)

5.5 Determining rules for graphs of exponential and logarithmic functions

In previous chapters, we considered establishing rules for graphs of some functions. In this chapter, we consider similar questions for exponential and logarithmic functions.

Example 15

The rule for the function of the graph is of the form \( y = ae^x + b \). Find the values of \( a \) and \( b \).
Solution
When \( x = 3 \), \( y = 22 \) and when \( x = 0 \), \( y = 6 \)
\[ \therefore 6 = ae^0 + b \quad (1) \]
and \[ 22 = ae^3 + b \quad (2) \]
Subtract (1) from (2):
\[ 16 = a(e^3 - e^0) \]
\[ \therefore 16 = a(e^3 - 1) \]
Therefore \( a = \frac{16}{e^3 - 1} \)
\[ \approx 0.8383 \]
From equation (1), \( b = 6 - a \)
\[ = 6 - \frac{16}{e^3 - 1} \]
\[ = \frac{6e^3 - 22}{e^3 - 1} \]
\[ \approx 5.1617 \]
Therefore \( y \approx 0.8383e^x + 5.1617 \)

Example 16
The rule for the function of the graph shown is of the form \( y = a\log_e(x + b) \). Find the values of \( a \) and \( b \).

Solution
When \( x = 5 \), \( y = 0 \) and when \( x = 8 \), \( y = 1 \)
\[ \therefore 0 = a\log_e(5 + b) \quad (1) \]
and \[ 1 = a\log_e(8 + b) \quad (2) \]
From (1) \( \log_e(5 + b) = 0 \)
\[ \therefore 5 + b = e^0 \]
and \( b = -4 \)
From (2) \[ 1 = a\log_e 4 \]
\[ \therefore a = \frac{1}{\log_e 4} \]
\[ \approx 0.7213 \]
\[ \therefore y \approx 0.7213\log_e(x - 4) \]

Example 17
Given that \( y = Ae^{bt} \) and \( y = 6 \) when \( t = 1 \) and \( y = 8 \) when \( t = 2 \), find the values of \( b \) and \( A \).
Solution

When \( t = 1 \), \( y = 6 \)
Thus \( 6 = Ae^b \)  \( (1) \)

When \( t = 2 \), \( y = 8 \)
Thus \( 8 = Ae^{2b} \)  \( (2) \)

Divide (2) by (1):
\[
\frac{4}{3} = e^b
\]
\[ ∴ b = \log_e \frac{4}{3} \]

Substitute in (1):
\[
6 = A e^{\log_e \frac{4}{3}}
\]
\[ ∴ 6 = \frac{4}{3} A \]
\[ ∴ A = \frac{18}{4} = \frac{9}{2} \]

Hence \( y = \frac{9}{2} e^{\log_e \frac{4}{3}} t \)
\[ = \frac{9}{2} \left( \frac{4}{3} \right)^t \]
\[ y \approx \frac{9}{2} e^{0.288t} \]

Using the TI-Nspire

The simultaneous equations template (\( ax + by = c \)) has been used here, but the result could also be obtained using solve \((6 = a \times e^b \textbf{ and } 8 = a \times e^{2b}, a, b)\).

\( \ln(x) = \log_e(x) \), the logarithm with base \( e \), is available on the keypad by pressing \( \ln \). Logarithms with other bases are obtained by pressing the log key (\( \log \)) and completing the template.

\textbf{Note:} Do not use the ‘\( e \)’ from the alpha keys. It will be treated as a variable.

Using the Casio ClassPad

Using the simultaneous equations template complete as shown.
Exercise 5E

1. The graph shown has rule:
   \[ y = ae^x + b \]
   Find the values of \( a \) and \( b \).

2. The rule for the function for which the graph is shown is of the form:
   \[ y = ae^x + b \]
   Find the values of \( a \) and \( b \).

3. The rule for the function \( f \) is of the form:
   \[ f(x) = ae^{-x} + b \]
   Find the values of \( a \) and \( b \).

4. Find the values of \( a \) and \( b \) such that the graph of \( y = ae^{-bx} \) goes through the points \((3, 50)\) and \((6, 10)\).

5. The rule of the graph shown is of the form:
   \[ y = a \log_2 (x - b) \]
   Find the values of \( a \) and \( b \).

6. Find the values of \( a \) and \( b \) such that the graph of \( y = ae^{bx} \) goes through the points \((3, 10)\) and \((6, 50)\).

7. Find the values of \( a \) and \( b \) such that the graph of \( y = a \log_2 x + b \) goes through the points \((8, 10)\) and \((32, 14)\).

8. Find the values of \( a \) and \( b \) such that the graph of \( y = a \log_2 (x - b) \) passes through the points \((5, 2)\) and \((7, 4)\).
9 The points (3, 10) and (5, 12) lie on the graph of the function with rule
\[ y = a \log_e (x - b) + c. \] The graph has a vertical asymptote with equation \( x = 1 \). Find the values of \( a, b \) and \( c \).

10 The graph of the function with rule \( f(x) = a \log_e (-x) + b \) passes through the points \((-2, 6)\) and \((-4, 8)\). Find the values of \( a \) and \( b \).

5.6 Change of base and solution of exponential equations

It is often useful to change the base of an exponential or logarithmic function, particularly to base 10 or \( e \) since these are the only ones available on the calculator.

To change the base of \( \log_a x \) from \( a \) to \( b \) (\( a > 0 \) and \( b > 0 \) and \( a, b \neq 1 \)), we use the definition that \( y = \log_a x \) implies \( a^y = x \).

Taking \( \log_b \) of both sides: \( \log_b a^y = \log_b x \)

Therefore \( y \log_b a = \log_b x \)

Therefore \( y = \frac{1}{\log_b a} \log_b x \)

Since \( y = \log_a x \)

\[ \log_a x = \frac{1}{\log_b a} \log_b x \]
or \[ \log_a x = \frac{\log_b x}{\log_b a} \]

This demonstrates that the graph of \( y = \log_a x \) can be obtained from the graph of \( y = \log_b x \) by a dilation of factor \( \frac{1}{\log_b a} \) from the \( x \)-axis.

A similar process shows that \( y = a^x \) can be written as \( x \log_b a = \log_b y \).

Rearranging to make \( y \) the subject: \( y = b^{(\log_b a)x} \)

Since \( y = a^x \):

\[ a^x = b^{(\log_b a)x} \]

This demonstrates that the graph of \( y = a^x \) can be obtained from the graph of \( y = b^x \) by a dilation of factor \( \frac{1}{\log_b a} \) from the \( y \)-axis.

The statement \( \log_a x = \frac{\log_b x}{\log_b a} \) can be used to simplify expressions, as in the following examples.
Example 18

Simplify:

\[ \frac{\log_e 27}{\log_e 3} \quad \text{a} \]
\[ \frac{\log_2 1024}{\log_2 4} \quad \text{b} \]

Solution

\[ \frac{\log_e 27}{\log_e 3} = \log_3 27 = 3 \]
\[ \frac{\log_2 1024}{\log_2 4} = \log_4 1024 = 5 \]

Example 19

Evaluate, correct to four significant figures:

\[ \log_2 10 \quad \text{a} \]
\[ \log_{\frac{1}{2}} 6 \quad \text{b} \]

Solution

\[ \log_2 10 = \frac{\log_e 10}{\log_e 2} \approx 3.322 \]
\[ \log_{\frac{1}{2}} 6 = \frac{\log_{10} 6}{\log_{10} \left(\frac{1}{2}\right)} \approx -2.585 \]

Example 20

If \( \log_2 6 = k \log_2 3 + 1 \), find the value of \( k \).

Solution

\[ \log_2 6 = k \log_2 (3) + 1 \]
\[ = \log_2 3^k + \log_2 2 \]
\[ = \log_2 (2 \times 3^k) \]

Therefore

\[ 6 = 2 \times 3^k \]
\[ 3 = 3^k \]
\[ k = 1 \]

Example 21

Solve for \( x \) if \( 2^x = 11 \), expressing the answer to two decimal places.

Solution

Take either the \( \log_{10} \) or \( \log_e \) (since these are the only logarithmic functions available on your calculator) of both sides of the equation:

Therefore

\[ \log_{10} 2^x = \log_{10} 11 \]

i.e.

\[ x \log_{10} 2 = \log_{10} 11 \]

Therefore

\[ x = \frac{\log_{10} 11}{\log_{10} 2} \approx 3.46 \]
Chapter 5 — Exponential and logarithmic functions

Example 22

Solve $3^{2x-1} = 28$, expressing the answer to three decimal places.

Solution

$$\log_e 3^{2x-1} = \log_e 28$$

$$2x - 1 = \frac{\log_e 28}{\log_e 3}$$

$$2x = \frac{\log_e 28}{\log_e 3} + 1$$

$$x = \frac{1}{2} \left( \frac{\log_e 28}{\log_e 3} + 1 \right)$$

$$\approx 2.017$$

Example 23

Solve \{ $x : 0.7^x \geq 0.3$ \}, expressing the answer to three decimal places.

Solution

Taking log_{10} of both sides:

$$\log_{10} 0.7^x \geq \log_{10} 0.3$$

$$x \log_{10} 0.7 \geq \log_{10} 0.3$$

$$x \leq \frac{\log_{10} 0.3}{\log_{10} 0.7}$$

(Note the sign change.)

$$x \leq -0.5229$$

$$x \leq -0.1549$$

$$x \leq 3.376$$

Exercise 5F

1. Use your calculator to solve each of the following equations, correct to two decimal places:

   a) $2^x = 6$
   b) $3^x = 0.7$
   c) $3^x = 11$
   d) $4^x = 5$
   e) $2^{-x} = 5$
   f) $0.2^x = 3$
   g) $5^x = 3^{x-1}$
   h) $8^x = 2005^{x+1}$
   i) $3^{x-1} = 8$
   j) $0.3^{x+2} = 0.7$
   k) $2^{x-1} = 3^{x+1}$
   l) $1.4^{x+2} = 25(0.9)^x$
   m) $5^x = 2^{2x-2}$
   n) $2^{(x+2)} = 3^{x-1}$
   o) $2^{x+1} \times 3^{x-1} = 100$

2. Solve for $x$ using a calculator. Express your answer correct to two decimal places.

   a) $2^x < 7$
   b) $3^x > 6$
   c) $0.2^x > 3$
   d) $3^{x-2} \leq 8$
   e) $0.2^x \leq 0.4$
3. Solve each of the following equations for $x$:
   \begin{align*}
   a) \quad 2^x &= 5 \\
   b) \quad 3^x &= 7 \\
   c) \quad 3^{2x} - 3^{x+2} + 8 &= 0
   \end{align*}

4. Simplify:
   \begin{align*}
   a) \quad \frac{\log_{10} 1024}{\log_{10} 2} \\
   b) \quad \frac{\log_5 216}{\log_5 6} \\
   c) \quad \frac{\log_4 81}{\log_4 3} + \frac{\log_4 256}{\log_4 3}
   \end{align*}

5. Evaluate, correct to four decimal places:
   \begin{align*}
   a) \quad \log_3 26 \\
   b) \quad \log_5 72 \\
   c) \quad \log_4 18 \\
   d) \quad \log_9 99 \\
   e) \quad \log_2 57 \\
   f) \quad \log_5 67 + \log_5 \left( \frac{1}{27} \right)
   \end{align*}

6. a) If $a \log_2 7 = 3 - \log_6 14$, find the value of $a$, correct to three significant figures.
    b) If $\log_3 18 = \log_{11} k$, find the value of $k$, correct to one decimal place.

7. Prove that $\log_a a + \log_e b + \log_a c = \frac{1}{\log_a b} + \frac{1}{\log_b c} + \frac{1}{\log_c a}$

8. Prove that if $\log_p p = q$ and $\log_q r = p$, then $\log_q p = pq$

9. If $u = \log_9 x$, find in terms of $u$:
   \begin{align*}
   a) \quad x & \\
   b) \quad \log_9 (3x) \\
   c) \quad \log_x 81
   \end{align*}

10. Solve the equation $\log_5 x = 16 \log_e 5$

11. Given that $q^p = 25$, find $\log_5 q$ in terms of $p$.

5.7 Inverses
It has been observed that $f(x) = \log_a x$ and $g(x) = a^x$ are inverse functions. In this section
this observation is used to find inverses of related functions, and to transform equations. An
important consequence is the following:
\begin{align*}
\log_a a^x &= x & \text{for all } x \in \mathbb{R} \\
\log_a x &= \log_a (x^a) & \text{for } x \in \mathbb{R}^+
\end{align*}

**Example 24**

Find the inverse of the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x + 2$ and state the domain and range of
the inverse function.

**Solution**
Recall that the transformation ‘a reflection in the line $y = x$’ is given by the mapping $(x, y) \to (y, x)$.

Consider $x = e^y + 2$
Then $x - 2 = e^y$
and $y = \log_e (x - 2)$
i.e., the inverse function has rule $f^{-1}(x) = \log_e (x - 2)$
The domain of $f^{-1}$ is the range of $f = (2, \infty)$. The range of $f^{-1}$ is the domain of $f = \mathbb{R}$. 

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Example 25

Rewrite the equation \( y = 2 \log_e(x) + 3 \) with \( x \) as the subject.

Solution

\[ y = 2 \log_e(x) + 3 \]

Therefore \( \frac{y - 3}{2} = \log_e x \)

and \( x = e^{\frac{y-3}{2}} \)

Example 26

Find the inverse of the function \( f: (1, \infty) \rightarrow R, f(x) = 2 \log_e(x - 1) + 3 \). State the domain and range of the inverse.

Solution

Consider \( x = 2 \log_e(y - 1) + 3 \)

Therefore \( \frac{x - 3}{2} = \log_e(y - 1) \)

and \( y - 1 = e^{\frac{x-3}{2}} \)

Therefore \( y = e^{\frac{x-3}{2}} + 1 \)

Hence \( f^{-1}(x) = e^{\frac{x-3}{2}} + 1 \)

The domain of \( f^{-1} \) is the range of \( f \)

\( f^{-1} = R \)

The range of \( f^{-1} \) is \((1, \infty)\).

Using the TI-Nspire

Use \textbf{Solve( )} from the \textbf{Algebra} menu as shown.
Using the Casio ClassPad
Enter and highlight $x = 2 \ln(y - 1) + 3$ then tap Interactive–Equation/inequality–solve and ensure the variable is set to $y$.

Example 27
Rewrite the equation $P = Ae^{kt}$ with $t$ as the subject.

Solution

\[ P = Ae^{kt} \]

Taking logarithms to the base $e$ of both sides:

\[ \log_e P = \log_e (Ae^{kt}) \]

\[ \therefore \log_e P = \log_e A + \log_e e^{kt} \]

\[ \therefore t = \frac{1}{k}(\log_e P - \log_e A) \]

\[ = \frac{1}{k} \log_e \left( \frac{P}{A} \right) \]

Exercise 5G
1. On the one set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ where $f: R \rightarrow R$, $f(x) = e^{-x} + 3$

2. On the one set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ where:
   $f: (1, \infty) \rightarrow R$, $f(x) = \log_e (x - 1)$

3. Find the inverse of each of the following functions and state the domain and range in each case:
   a. $f: R^+ \rightarrow R$, where $f(x) = \log_e 2x$
   b. $f: R^+ \rightarrow R$, where $f(x) = 3 \log_e (2x) + 1$
   c. $f: R \rightarrow R$, where $f(x) = e^x + 2$
   d. $f: R \rightarrow R$, where $f(x) = e^{x^2}$
   e. $f: \left( -\frac{1}{2}, \infty \right) \rightarrow R$, where $f(x) = \log_e (2x + 1)$
   f. $f: \left( -\frac{2}{3}, \infty \right) \rightarrow R$, where $f(x) = 4 \log_e (3x + 2)$
   g. $f: \{x: x > -1\} \rightarrow R$, $f(x) = \log_{10} (x + 1)$
   h. $f: R \rightarrow R$, $f(x) = 2e^{x-1}$
4 The function \( f \) has the rule \( f(x) = 1 - e^{-x} \)
   a Sketch the graph of \( f \).
   b Find the domain of \( f^{-1} \) and \( f^{-1}(x) \).
   c Sketch the graph of \( f^{-1} \) on the same set of axes as the graph of \( f \).

5 Let \( f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 5e^{2x} - 3 \)
   a Sketch the graph of \( f \).
   b Find the inverse function \( f^{-1} \).
   c Sketch the graph of \( f^{-1} \) on the same set of axes as the graph of \( f \).

6 Let \( f: \mathbb{R}^+ \rightarrow \mathbb{R}, f(x) = 2 \log_e(x) + 1 \)
   a Sketch the graph of \( f \).
   b Find the inverse function \( f^{-1} \) and state the range.
   c Sketch the graph of \( f^{-1} \) on the same set of axes as the graph of \( f \).

7 For each of the formulas, make the pronumeral in brackets the subject:
   a \( y = 2 \log_e(x) + 5 \) \( (x) \)
   b \( P = Ae^{-6x} \) \( (x) \)
   c \( y = ax^n \) \( (n) \)
   d \( y = 5 \times 10^x \) \( (x) \)
   e \( y = 5 - 3 \log_e(2x) \) \( (x) \)
   f \( y = 6x^2n \) \( (n) \)
   g \( y = \log_e(2x - 1) \) \( (x) \)
   h \( y = 5(1 - e^{-x}) \) \( (x) \)

8 For \( f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2e^x - 4 \):
   a Find the inverse function \( f^{-1} \).
   b Find the coordinates of the points of intersection of the graphs of \( y = f(x) \) and \( y = f^{-1}(x) \)

9 For \( f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2 \log_e(x + 3) + 4 \):
   a Find the inverse function \( f^{-1} \).
   b Find the coordinates of the points of intersection of the graphs of \( y = f(x) \) and \( y = f^{-1}(x) \)

10 a Using a calculator, for each of the following plot the graphs of \( y = f(x) \) and \( y = g(x) \), together with the line \( y = x \), on the one set of axes.
   i \( f(x) = \log_e x \) and \( g(x) = e^x \)
   ii \( f(x) = 2 \log_e(x) + 3 \) and \( g(x) = e^{x-3} \)
   iii \( f(x) = \log_{10} x \) and \( g(x) = 10^x \)
   b Use your answers to part a to comment on the relationship between \( f(x) = a \log_b x + c \) and \( g(x) = b \frac{x-c}{a} \).
5.8 Exponential growth and decay

Exponential and logarithmic functions are used to model many physical occurrences. It will be shown in Chapter 11 that if a quantity increases or decreases at a rate which is, at any time, proportional to the quantity present, then the quantity present at time \( t \) is given by the law of exponential change.

Let \( A \) be the quantity at time \( t \). Then, \( A = A_0 e^{kt} \), where \( A_0 \) is a constant.

- Growth: \( k > 0 \)
- Decay: \( k < 0 \)

The number \( k \) is the rate constant of the equation.

Physical situations where this is applicable include:
- the growth of a cell
- population growth
- continuously compounded interest
- radioactive decay
- Newton’s law of cooling.

Example 28

A bank pays 10% interest compounded annually. You invest $1000. How does this $1000 grow as a result of the interest added?

**Solution**

Set out in tabular form.

<table>
<thead>
<tr>
<th>End of year</th>
<th>Amount, ( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1000</td>
<td>$1000.00</td>
</tr>
<tr>
<td>1</td>
<td>[ 1000 \times 110% = 1000 \left( 1 + \frac{10}{100} \right) = 1100.00 ]</td>
</tr>
<tr>
<td>2</td>
<td>[ (1000 \times 110%) = 1000 \left( 1 + \frac{10}{100} \right)^2 = 1210.00 ]</td>
</tr>
<tr>
<td>3</td>
<td>[ (1000 \times 110%) = 1000 \left( 1 + \frac{10}{100} \right)^3 = 1331.00 ]</td>
</tr>
<tr>
<td>4</td>
<td>[ = 1000 \left( 1 + \frac{10}{100} \right)^4 = 1464.10 ]</td>
</tr>
<tr>
<td>↓</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>[ = 1000 \left( 1 + \frac{10}{100} \right)^{10} = 2593.74 ]</td>
</tr>
<tr>
<td>↓</td>
<td></td>
</tr>
<tr>
<td>( n )</td>
<td>[ = 1000 \left( 1 + \frac{10}{100} \right)^n = 1000 (1.1)^n ]</td>
</tr>
</tbody>
</table>
Chapter 5 — Exponential and logarithmic functions

A = 1000(1.1)^n, n ∈ \mathbb{N} \cup \{0\}

In general, if \( P \) = original investment
\[ A = \text{amount the investment grows to after } n \text{ years} \]
\( r = \text{compound interest rate } r\% \text{ per annum} \)
\( n = \text{number of years invested} \)

then \[ A = P \left(1 + \frac{r}{100}\right)^n \]

Example 29

The population of a town was 8000 at the beginning of 1992 and 15 000 at the end of 1999. Assume that the growth is exponential.

a Find the population at the end of 2001.

b In what year will the population be double that of 1999?

Solution

a Let \( P \) be the population at time \( t \) years (measured from 1 January 1992). Then \[ P = 8000e^{kt} \]

At the end of 1999, \( t = 8 \) and \( P = 15 000 \).

\[ \therefore 15 000 = 8000e^{8k} \]

\[ \therefore \frac{15}{8} = e^{8k} \]

\[ \therefore k = \frac{1}{8} \log_e \frac{15}{8} \]

\[ \approx 0.079 \]

The rate of increase is 7.9% per annum.

Note: The approximation 0.079 was not used in the calculations which follow. The value for \( k \) was held in the calculator.

When \( t = 10 \)

\[ P = 8000e^{10k} \]

\[ \approx 17 552.6049 \]

\[ \approx 17 550 \]

The population is approximately 17 550.

b When does \( P = 30 000 \)? Consider the equation:

\[ 30 000 = 8000e^{8k} \]

\[ \therefore \frac{30 000}{8000} = e^{8k} \]

\[ \therefore \frac{15}{4} = e^{8k} \]

\[ \therefore 3.75 = e^{8k} \]

\[ \therefore t = \frac{1}{k} \log_e 3.75 \]

\[ \approx 16.82 \]
The population reaches 30,000 approximately 16.82 years after the beginning of 1992, i.e. during the year 2008.

Exercise 5H

1 In the initial period of its life a particular species of tree grows in the manner described by the rule \( d = d_0 10^m t \) where \( d \) is the diameter of the tree in centimetres, \( t \) years after the beginning of this period. The diameter after 1 year is 52 cm and after 3 years, 80 cm. Calculate the values of the constants \( d_0 \) and \( m \).

2 The number of bacteria in a certain culture at time, \( t \) weeks, is given by the rule \( N = N_0 e^{kt} \). If when \( t = 2 \), \( N = 101 \) and when \( t = 4 \), \( N = 203 \) calculate the values of \( N_0 \) and \( k \).

3 The number of people, \( N \), who have a particular disease at time \( t \) years is given by \( N = N_0 e^{kt} \)
   a If the number initially is 20,000 and the number decreases by 20% each year, find:
      i the value of \( N_0 \) ii the value of \( k \)
   b How long does it take for 5000 people to be infected?

4 Polonium-210 is a radioactive substance. The decay of polonium-210 is described by the formula \( M = M_0 e^{-kt} \), where \( M \) is the mass in grams of polonium-210 left after \( t \) days, and \( M_0 \) and \( k \) are constants. At time \( t = 0 \), \( M = 10 \) g and at \( t = 140 \), \( M = 5 \) g.
   a Find the values of \( M_0 \) and \( k \).
   b What will be the mass of the polonium-210 after 70 days?
   c After how many days is the mass remaining 2 g?
Chapter summary

- **Sketch graphs** of \( y = a^x \), e.g. \( a = 2 \) or 10 or \( e \), and transformations of these graphs:

- **Index laws**
  
  \[
  a^m \times a^n = a^{m+n} \\
  a^m \div a^n = a^{m-n} \\
  (a^m)^n = a^{mn}
  \]

- **Logarithms**
  
  \( \log_a x = y \) if \( a^y = x \)

  The inverse function of \( f: R \to R, f(x) = a^x \) is \( f^{-1}: R^+ \to R, f^{-1}(x) = \log_a x \)

- **Sketch graphs** of \( y = \log_a x \), e.g. \( a = 2 \) or 10 or \( e \), and transformations of these graphs:

- **Logarithm laws**
  
  \[
  \log_a (mn) = \log_a m + \log_a n \\
  \log_a \left( \frac{m}{n} \right) = \log_a m - \log_a n \\
  \log_a \left( \frac{1}{n} \right) = -\log_a n \\
  \log_a (m^n) = p \log_a m
  \]

- **Change of base**
  
  \[
  \log_a x = \frac{\log_b x}{\log_b a} \\
  a^x = b^{(\log_b a)x}
  \]

- **Inverse properties**
  
  \[
  \log_a a^x = x \\
  \text{and} \quad a^{\log_a x} = x
  \]
Law of exponential change

Let \( A \) be the quantity at time \( t \). Then:

\[
A = A_0 e^{kt}, \quad \text{where } A_0 \text{ is a constant.}
\]

Growth: \( k > 0 \)  Decay \( k < 0 \)

The number \( k \) is the rate constant of the equation.

Multiple-choice questions

1. If \( 4 \log_b |x| = \log_b 16 + 8 \), then \( x \) is equal to:
   - A 24
   - B \( \pm 6 \)
   - C \( \pm 2b^2 \)
   - D 6
   - E \( 2b^2 \)

2. The expression \( \log_e (4e^{3x}) \) is equal to:
   - A \( \log_e (e^{12x}) \)
   - B \( \log_e 12 + x \)
   - C \( 3x \log_e 4 \)
   - D \( \log_e 4 + 3x \)
   - E \( 12x \)

3. The expression \( 3^{\log_3(x-4)} \) is equal to:
   - A \( \frac{x}{4} \)
   - B \( x - 4 \)
   - C \( 3(x - 4) \)
   - D \( 3^x - 3^4 \)
   - E \( \log_3 x - \log_3 4 \)

4. Let \( f: A \to R, f(x) = e^{2x}, g: B \to R, g(x) = \frac{1}{x + 1} \), and \( h: C \to R, h(x) = e^{2x} + \frac{1}{x + 1} \), where \( A, B \) and \( C \) are the largest domains for which \( f, g \) and \( h \) respectively are defined. Which one of the following statements is true?
   - A \( A \neq C \) and \( \text{ran } g = \text{ran } h \)
   - B \( A = B \) and \( \text{ran } f \neq \text{ran } h \)
   - C \( A \neq C \) and \( \text{ran } f = \text{ran } h \)
   - D \( B = C \) and \( \text{ran } g = \text{ran } h \)
   - E \( B = C \) and \( \text{ran } g \neq \text{ran } h \)

5. If \( x = 5 \) is a solution of the equation \( \log_{10} (kx - 3) = 2 \), then the exact value of \( k \) is:
   - A \( \frac{103}{5} \)
   - B \( \frac{\log_{10}^2 + 3}{5} \)
   - C 2
   - D 5
   - E 21

6. \( 3^{(4\log_3 x + \log_3 4x)} \) is equal to:
   - A \( 8x \)
   - B \( x^4 + 4x \)
   - C \( 4x^5 \)
   - D \( 3^{8x} \)
   - E \( \log_3 4x^5 \)

7. The solution of the equation \( 3x = 10^{-0.3x} \) is closest to:
   - A 0.83
   - B 0.28
   - C 0
   - D 0.30
   - E 0.91

8. The graph of the function with equation \( y = ae^{-x} + b \) is shown below.

   ![Graph](image)

   The values of \( a \) and \( b \) respectively are:
   - A 3, -3
   - B -3, 3
   - C -3, -3
   - D 0, -3
   - E -3, 0
For the function \( f: R^+ \to R, f(x) = \log_5 x \):

9. Which one of the following statements is not true of the graph of the function \( f: R^+ \to R, f(x) = \log_5 x \)?

- A. The domain is \( R^+ \).
- B. The range is \( R \).
- C. It passes through the point \((5, 0)\).
- D. It has a vertical asymptote with equation \( x = 0 \).
- E. The slope of the tangent at any point on the graph is positive.

10. If \( 3 \log_2 x - 7 \log_2 (x - 1) = 2 + \log_2 y \), then \( y \) is equal to:

\[
\begin{array}{cc}
A & \frac{3x}{28(x - 1)} \\
B & \frac{1}{4x^4} \\
C & 3 - 4x \\
D & \frac{4(x - 1)^2}{4(x - 1)} \\
E & x^3 - (x - 1)^7 - 4
\end{array}
\]

### Short-answer questions (technology-free)

1. Sketch the graphs of each of the following. Label asymptotes and axes intercepts.

   \( f(x) = e^x - 2 \)

   \( g(x) = 10^{-x} + 1 \)

   \( h(x) = \frac{1}{2}(e^x - 1) \)

   \( f(x) = 2 - e^{-x} \)

   \( f(x) = \log_e (2x + 1) \)

   \( f(x) = \log_e (x - 1) + 1 \)

   \( g(x) = -\log_e (x - 1) \)

   \( f(x) = -\log_e (1 - x) \)

2. a. For \( f: R \to R, f(x) = e^{2x} - 1 \), find \( f^{-1} \).
   
   b. For \( f: (2, \infty) \to R, f(x) = 3 \log_e (x - 2) \), find \( f^{-1} \).
   
   c. For \( f: (-1, \infty) \to R, f(x) = \log_{10} (x + 1) \), find \( f^{-1} \).
   
   d. For \( f: R^+ \to R, f(x) = 2^x + 1 \), find \( f^{-1} \).

3. For each of the following, find \( y \) in terms of \( x \):

   \( \log_e y = (\log_e x) + 2 \)

   \( \log_{10} y = \log_{10} x + 1 \)

   \( \log_2 y = 3 \log_2 x + 4 \)

   \( \log_{10} y = -1 + 5 \log_{10} x \)

   \( \log_e y = 3 - \log_e x \)

   \( \log_e y = 2x - 3 \)

4. Solve each of the following equations for \( x \), expressing your answers in terms of logarithms of base \( e \):

   \( 3^x = 11 \)

   \( 2^x = 0.8 \)

   \( 2^x = 3^{x+1} \)

5. Solve each of the following for \( x \):

   \( 2^{2x} - 2^x - 2 = 0 \)

   \( \log_e (3x - 1) = 0 \)

   \( \log_{10} (2x) + 1 = 0 \)

   \( 10^{2x} - 7 \times 10^x + 12 = 0 \)

6. The graph of the function with equation \( y = 3 \log_2 (x + 1) + 2 \) intersects the axes at the points \((a, 0)\) and \((0, b)\). Find the exact values of \( a \) and \( b \).

7. The graph of the function with equation \( f(x) = 5 \log_{10} (x + 1) \) passes through the point \((k, 6)\). Find the value of \( k \).

8. Find the exact value of \( x \) for which \( 4e^{3x} = 287 \).

9. Find the value of \( x \) in terms of \( a \) for which \( 3 \log_a x = 3 + \log_a 8 \).

10. For the function \( f: (4, \infty) \to R, f(x) = \log_3 (x - 4) \), state the domain of the inverse function \( f^{-1} \).

11. The graph of the function with equation \( f(x) = e^{2x} - 3ke^x + 5 \) intersects the axes at \((0, 0)\) and \((a, 0)\) and has a horizontal asymptote at \( y = b \). Find the exact values of \( a, b \) and \( k \).
Extended-response questions

1. A liquid cools from its original temperature of $90^\circ \text{C}$ to a temperature $T^\circ \text{C}$ in $x$ minutes. Given that $T = 90(0.98)^x$, find:
   a. the value of $T$ when $x = 10$
   b. the value of $x$ when $T = 27$

2. The population of a village at the beginning of the year 1800 was 240. The population increased so that, after a period of $n$ years, the new population was $240(1.06)^n$. Find:
   a. the population at the beginning of 1820
   b. the year in which the population first reached 2500

3. The value, $V$, of a particular car can be modelled by the equation:

   $$V = ke^{-\lambda t}$$

   where $t$ years is the age of the car.

   The car’s original price was $22 497, and after 1 year it is valued at $18 000.
   a. State the value of $k$ and calculate $\lambda$, giving your answer to two decimal places.
   b. Find the value of the car when it is 3 years old.

4. The value $M$ of a particular house in a certain area during the period 1988 to 1994 can be modelled by the equation $M = Ae^{-pt}$ where $t$ is the time in years after 1 January 1988.

   The value of the house on 1 January 1988 was $65 000 and its value on 1 January 1989 was $61 000.
   a. State the value of $A$ and calculate the value of $p$, correct to two significant figures.
   b. What was the value of the house in 1993? Give your answer to the nearest 100.

5. There are two species of insects living in a suburb: the *Asla bibla* and the *Cutus pius*. The number of *Asla bibla* alive at time $t$ days after 1 January 2000 is given by:

   $$N_A(t) = 10 000 + 1000t \quad 0 \leq t \leq 15$$

   The number of *Cutus pius* alive at time $t$ days after 1 January 2000 is given by:

   $$N_C(t) = 8000 + 3 \times 2^t \quad 0 \leq t \leq 15$$

   a. With a calculator plot the graphs of $y = N_A(t)$ and $y = N_C(t)$ on the one screen.
   b. i. Find the coordinates of the point of intersection of the two graphs.
      ii. At what time is $N_C(t) = N_A(t)$?
      iii. What is the number of each species of insect at this time?
   c. i. Show that $N_A(t) = N_C(t)$ if and only if $t = \frac{1}{\log_{10} 2} \left[ 3 + \log_{10} \left( \frac{2 + t}{3} \right) \right]$.
      ii. Plot the graphs of $y = x$ and $y = \frac{1}{\log_{10} 2} \left[ 3 + \log_{10} \left( \frac{2 + x}{3} \right) \right]$ and find the coordinates of the point of intersection.
It is found by observation that the model for *Cutus pius* doesn’t quite work. It is known that the model for the population of *Asla bibla* is satisfactory. The form of the model for *Cutus pius* is $N_c(t) = 8000 + c \times 2^t$.

Find the value of $c$, correct to two decimal places, if it is known that $N_A(15) = N_C(15)$.

The number of a type of bacteria is modelled by the formula $n = A(1 - e^{-Bt})$ where $n$ is the size of the population at time $t$ hours. $A$ and $B$ are positive constants.

a When $t = 2$, $n = 10000$ and when $t = 4$, $n = 15000$.
   i Show that $2e^{-4B} - 3e^{-2B} + 1 = 0$.
   ii Use the substitution $a = e^{-2B}$ to show that $2a^2 - 3a + 1 = 0$.
   iii Solve this equation for $a$.
   iv Find the exact value of $B$.
   v Find the exact value of $A$.

Sketch the graph of $n$ against $t$.

b After how many hours is the population of bacteria 18 000?

d A radioactive substance is decaying such that the amount $A$ g at time $t$ years is given by the formula $A = A_0e^{kt}$. When $t = 1$, $A = 60.7$ and when $t = 6$, $A = 5$, find the values of the constants $A_0$ and $k$.

e In a chemical reaction the amount (x g) of a substance that has reacted is given by:
   $$x = 8(1 - e^{-0.2t})$$
   where $t$ is the time from the beginning of the reaction, in minutes.
   a Sketch the graph of $x$ against $t$.
   b Find the amount of substance that has reacted after:
      i 0 minutes  ii 2 minutes  iii 10 minutes
   c Find the time when exactly 7 g of the substance has reacted.

e A radioactive substance is decaying such that the amount $A$ g at time $t$ years is given by the formula $A = A_0e^{kt}$. When $t = 1$, $A = 60.7$ and when $t = 6$, $A = 5$, find the values of the constants $A_0$ and $k$.

Newton’s law of cooling for a body placed in a medium of constant temperature states:
$$T - T_s = (T_0 - T_s)e^{-kt}$$
where:
$T$ is the temperature (in °C) of the body at time $t$ (in minutes)
$T_s$ is the temperature of the surrounding medium, and
$T_0$ is the initial temperature of the body.

An egg at 96°C is placed in a sink of water at 15°C to cool. After 5 minutes the egg’s temperature is found to be 40°C. (Assume that the temperature of the water does not change.)

a Find the value of $k$.

b Find the temperature of the egg when $t = 10$.

c How long does it take for the egg to reach a temperature of 30°C?
The population of a colony of small, interesting insects is modelled by the following hybrid function:

\[ N(t) = \begin{cases} 
20e^{0.2t} & 0 \leq t \leq 50 \\
20e^{10} & 50 \leq t \leq 70 \\
10e^{10(e^{70-t} + 1)} & t > 70 
\end{cases} \]

where \( t \) is the number of days.

a Sketch the graph of \( N(t) \) against \( t \).

b Find:

i \( N(10) \) ii \( N(40) \) iii \( N(60) \) iv \( N(80) \)

c Find the number of days for the population to reach:

i 2968 ii 21 932