CHAPTER 2

Revising linear functions and matrices

Objectives

- To revise:
  - methods for solving linear equations
  - methods for solving simultaneous linear equations
  - calculating the gradient of a straight line
  - interpreting and using the general equation of a straight line: \( y = mx + c \)
  - a method for determining the gradient of a line perpendicular to a given line
  - finding the distance between two points
  - finding the midpoint of a straight line
  - calculating the angle between two intersecting straight lines
  - matrix arithmetic

- To apply a knowledge of linear functions to solving problems.

It is assumed that the material in this chapter has been covered by students in *Essential Mathematical Methods 1 & 2*. The chapter provides a framework for revision with worked examples and practice exercises.

2.1 Linear equations

Exercise 2A

1. Solve the following linear equations:

   \[ \begin{align*}
   a & \quad 3x - 4 = 2x + 6 \\
   b & \quad 8x - 4 = 3x + 1 \\
   c & \quad 3(2 - x) - 4(3 - 2x) = 14 \\
   d & \quad \frac{3x}{4} - 4 = 17 \\
   e & \quad 6 - 3y = 5y - 62 \\
   f & \quad \frac{2}{3x - 1} = \frac{3}{7}
   \end{align*} \]
2 Solve each of the following pairs of simultaneous linear equations:

a \[ x - 4 = y \quad 4y - 2x = 8 \]

b \[ 9x + 4y = 13 \quad 2x + y = 2 \]

c \[ 7x = 18 + 3y \quad 2x + 5y = 11 \]

d \[ 5x + 3y = 13 \quad 19x + 17y = 0 \]

e \[ 7x + 2y = 16 \quad 2x - y = 53 \]

f \[ \frac{x}{5} + \frac{y}{2} = 5 \quad x - y = 4 \]

3 An aircraft, used for fire spotting, flies from its base to locate a fire at an unknown distance, \( x \) km away. It travels straight to the fire and back, averaging 240 km/h for the outward trip and 320 km/h for the return trip. If the plane was away for 35 minutes, find the distance, \( x \) km.

4 A group of hikers is to travel \( x \) km by bus at an average speed of 48 km/h to an unknown destination. They then plan to walk back along the same route at an average speed of 4.8 km/h and to arrive back 24 hours after setting out in the bus. If they allow 2 h for lunch and rest, how far must the bus take them?

5 The length of a rectangle is 4 cm more than the width. If the length were to be decreased by 5 cm and the width decreased by 2 cm, the perimeter would be 18 cm. Calculate the dimensions of the rectangle.

6 In a basketball game a field goal scores two points and a free throw scores one point. John scored 11 points in the game and David 19 points. David scored the same number of free throws as John but twice as many field goals. How many field goals did each score?

7 The weekly wage, \( w \), of a salesman consists of a fixed amount of $800 and then $20 for each unit he sells.

a If he sells \( n \) units a week, find a rule for his weekly wage, \( w \), in terms of the number of units sold.

b Find his wage if he sells 30 units.

c How many units does he sell if his weekly wage is $1620?

8 Water flows into a tank at a rate of 15 litres per minute. At the beginning the tank contained 250 litres.

a Write an expression for the volume, \( V \) litres, of water in the tank at time \( t \) minutes.

b How many litres of water are there in the tank after an hour?

c The tank has a capacity of 5000 litres. How long does it take to fill?

9 A tank contains 10 000 litres of water. Water flows out at a rate of 10 litres per minute.

a Write an expression for the volume, \( V \) litres, of water in the tank at time \( t \) minutes.

b How many litres of water are there in the tank after an hour?

c How long does it take for the tank to empty?
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10 The cost, \( C \), of hiring diving equipment is $100 plus $25 per hour.

a Write a rule which gives the total charge, \( C \), of hiring the equipment for \( t \) hours (assume that parts of hours are paid for proportionately).

b Find the cost of hiring the equipment for:
   i 2 hours
   ii 2 hours 30 minutes

c For how many hours can the equipment be hired if the following amounts are available?
   i $375
   ii $400

2.2 Linear literal equations and simultaneous linear literal equations

A literal equation in \( x \) is an equation whose solution will be expressed in terms of pronumerals rather than numbers.

\( 2x + 5 = 7 \) is an equation whose solution is the number 1.

In the literal equation \( ax + b = c \), the solution is \( x = \frac{c - b}{a} \).

Literal equations are solved in the same way as solving numerical equations or transposing formulas. Essentially, the literal equation is transposed to make \( x \) the subject.

**Example 1**

Solve the following for \( x \).

\[ a \]
\[ px - q = r \]

\[ b \]
\[ ax + b = cx + d \]

\[ c \]
\[ \frac{a}{x} = \frac{b}{2x} + c \]

**Solution**

\[ a \]
\[ px - q = r \]
\[ px = r + q \]
\[ x = \frac{r + q}{p} \]

\[ b \]
\[ ax + b = cx + d \]
\[ ax - cx = d - b \]
\[ x(a - c) = d - b \]
\[ x = \frac{d - b}{a - c} \]

\[ c \]
\[ \frac{a}{x} = \frac{b}{2x} + c \]

Multiply both sides of the equation by \( 2x \).

\[ 2a = b + 2xc \]
\[ 2a - b = 2xc \]
\[ 2a - b = \frac{2c}{2c} = x \]

**Simultaneous literal equations**

Simultaneous literal equations are solved by the methods of solution of simultaneous equations, i.e. substitution and elimination.
Solve each of the following pairs of simultaneous equations for $x$ and $y$.

a $y = ax + c$
$y = bx + d$

b $ax - y = c$
$x + by = d$

Solution

\[
\begin{align*}
\text{a} & \quad ax + c = bx + d \\
& \quad \therefore ax - bx = d - c \\
& \quad x(a - b) = d - c \\
& \quad x = \frac{d - c}{a - b}
\end{align*}
\]

and therefore

\[
\begin{align*}
y &= a \left( \frac{d - c}{a - b} \right) + c \\
&= \frac{ad - bc}{a - b}
\end{align*}
\]

\[
\begin{align*}
\text{b} & \quad ax - y = c \quad \ldots (1) \\
x + by = d \quad \ldots (2)
\end{align*}
\]

Multiply (1) by $b$:

\[
abx - by = cb \quad \ldots (1')
\]

Add (1') and (2):

\[
x(ab + 1) = cb + d \\
x = \frac{cb + d}{ab + 1}
\]

Substitute in (1):

\[
a \left( \frac{cb + d}{ab + 1} \right) - y = c
\]

\[
\therefore y = a \left( \frac{cb + d}{ab + 1} \right) - c
\]

\[
= \frac{ad - c}{ab + 1}
\]

Exercise 2B

1 Solve each of the following for $x$:

\[
\begin{align*}
\text{a} & \quad ax + n = m \\
\text{b} & \quad ax + b = bx
\end{align*}
\]

\[
\begin{align*}
\text{c} & \quad \frac{ax}{b} + c = 0 \\
\text{d} & \quad px = qx + 5
\end{align*}
\]

\[
\begin{align*}
\text{e} & \quad mx + n = nx - m \\
\text{f} & \quad \frac{1}{x + a} = \frac{b}{x}
\end{align*}
\]

\[
\begin{align*}
\text{g} & \quad \frac{b}{x - a} = \frac{2b}{x + a} \\
\text{h} & \quad \frac{x}{m} + n = \frac{x}{n} + m
\end{align*}
\]

\[
\begin{align*}
\text{i} & \quad -b(ax + b) = a(bx - a) \\
\text{j} & \quad p^2(1 - x) - 2pqx = q^2(1 + x)
\end{align*}
\]

\[
\begin{align*}
\text{k} & \quad \frac{x}{a} - 1 = \frac{x}{b} + 2 \\
\text{l} & \quad \frac{x}{a - b} + \frac{2x}{a + b} = \frac{1}{a^2 - b^2}
\end{align*}
\]

\[
\begin{align*}
\text{m} & \quad \frac{p - qx}{t} + p = \frac{qx - t}{p} \\
\text{n} & \quad \frac{1}{x + a} + \frac{1}{x + 2a} = \frac{2}{x + 3a}
\end{align*}
\]

2 For the simultaneous equations $ax + by = p$ and $bx - ay = q$, show that $x = \frac{ap + bq}{a^2 + b^2}$ and $y = \frac{bp - aq}{a^2 + b^2}$
3 For the simultaneous equations $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$, show that $x = y = \frac{ab}{a+b}$.

4 Solve each of the following pairs of simultaneous equations for $x$ and $y$.

- $ax + y = c$  
  - $x + by = d$
- $ax + by = t$  
  - $ax - by = s$
- $(a + b)x + cy = bc$  
  - $(b + c)y + ax = -ab$

5 Write $s$ in terms of $a$ only in the following pairs of equations:

- $s = ah$  
  - $h = 2a + 1$
- $as = a + h$  
  - $h + ah = 1$
- $s = h^2 + ah$  
  - $h = 3a^2$
- $s = 2 + ah + h^2$  
  - $h = a - \frac{1}{a}$

2.3 Linear coordinate geometry

The following is a summary of the material that is assumed to have been covered in Essential Mathematical Methods 1 & 2.

**Gradient of a straight line joining two points**

Gradient, $m = \frac{y_2 - y_1}{x_2 - x_1}$

**The general equation of a straight line**

$y = mx + c$

where $m$ is the gradient and $c$ is the value of the intercept on the $y$-axis.
Essential Mathematical Methods 3 & 4 CAS

Equation of a straight line passing through a given point \((x_1, y_1)\) and having a gradient, \(m\)

Equation of line is:

\[ y - y_1 = m(x - x_1) \]

Equation of a line passing through two given points \((x_1, y_1)\) and \((x_2, y_2)\)

Equation of line is:

\[ y - y_1 = m(x - x_1) \text{ where } m = \frac{y_2 - y_1}{x_2 - x_1} \]

The intercept form of the equation of a straight line

For a line passing through the points \((a, 0)\) and \((0, b)\), the equation is:

\[ \frac{x}{a} + \frac{y}{b} = 1 \]

Tangent of the angle of slope \((\theta)\)

\[ \tan \theta = \frac{y_2 - y_1}{x_2 - x_1} \]

where \(\theta\) is the angle the line makes with the positive direction of the \(x\)-axis.

Product of gradients of two perpendicular straight lines

If two straight lines are perpendicular to each other, the product of their gradients is \(-1\).

\[ m_1m_2 = -1 \]
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Distance between two points

\[ AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

Midpoint

The midpoint of a straight line joining \((x_1, y_1)\) and \((x_2, y_2)\) is the point \(\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\).

The angle \(\alpha\) between intersecting straight lines

\[ \alpha = \theta_2 - \theta_1 \]

Example 3

A fruit and vegetable wholesaler sells 30 kg of hydroponic tomatoes for $148.50 and 55 kg of hydroponic tomatoes for $247.50. Find a linear model for the cost, \(C\) dollars, to buy \(x\) kg of hydroponic tomatoes. How much would 20 kg of tomatoes cost?

Solution

Let \((x_1, C_1) = (30, 148.5)\) and \((x_2, C_2) = (55, 247.5)\).

The equation of the straight line is given by:

\[ C - C_1 = m(x - x_1) \quad \text{where} \quad m = \frac{C_2 - C_1}{x_2 - x_1} \]

Now \(m = \frac{247.5 - 148.5}{55 - 30} = 3.96\) so \(C - 148.5 = 3.96(x - 30)\)

Therefore the straight line is given by the equation \(C = 3.96x + 29.7\)

Substitute \(x = 20\):

\[ C = 3.96 \times 20 + 29.7 = 108.9 \]

It would cost $108.90 to buy 20 kg of tomatoes.

Exercise 2C

1. Find the coordinates of \(M\), the midpoint of \(AB\), where \(A\) and \(B\) have the following coordinates:
   
   a. \(A(1, 4), B(5, 11)\)  
   b. \(A(-6, 4), B(1, -8)\)  
   c. \(A(-1, -6), B(4, 7)\)
2 Use \( y = mx + c \) to sketch the graph of each of the following:

\[ \begin{align*}
\text{a} & : y = 3x - 3 \\
\text{b} & : y = -3x + 4 \\
\text{c} & : 3y + 2x = 12 \\
\text{d} & : 4x + 6y = 12 \\
\text{e} & : 3y - 6x = 18 \\
\text{f} & : 8x - 4y = 16
\end{align*} \]

3 Find the equations of the following straight lines:

\[ \begin{align*}
\text{a} & : \text{gradient} + 2, \text{passing through } (4, 2) \\
\text{b} & : \text{gradient} - 3, \text{passing through } (-3, 4) \\
\text{c} & : \text{passing through the points } (1, 3) \text{ and } (4, 7) \\
\text{d} & : \text{passing through the points } (-2, -3) \text{ and } (2, 5)
\end{align*} \]

4 Use the intercept method to find the equation of the straight lines passing through:

\[ \begin{align*}
\text{a} & : (-3, 0) \text{ and } (0, 2) \\
\text{b} & : (4, 0) \text{ and } (0, 6) \\
\text{c} & : (-4, 0) \text{ and } (0, -3) \\
\text{d} & : (0, -2) \text{ and } (6, 0)
\end{align*} \]

5 Write each of the following in intercept form and hence draw their graphs:

\[ \begin{align*}
\text{a} & : 3x + 6y = 12 \\
\text{b} & : 4y - 3x = 12 \\
\text{c} & : 4y - 2x = 8 \\
\text{d} & : \frac{3}{2}x - 3y = 9
\end{align*} \]

6 A printing firm charges \$35 for printing 600 sheets of headed notepaper and \$46 for printing 800 sheets. Find a linear model for the charge, \( C \) dollars, for printing \( n \) sheets. How much would they charge for printing 1000 sheets?

7 An electronic bank teller registered \$775 after it had counted 120 notes and \$975 after it had counted 160 notes.

\[ \begin{align*}
\text{a} & : \text{Find a formula for the sum registered, } (\$C), \text{ in terms of the number of notes } (n) \text{ counted.} \\
\text{b} & : \text{Was there a sum already on the register when counting began?} \\
\text{c} & : \text{If so, how much?}
\end{align*} \]

8 Find the distance between each of the following pairs of points:

\[ \begin{align*}
\text{a} & : (2, 6), (3, 4) \\
\text{b} & : (5, 1), (6, 2) \\
\text{c} & : (-1, 3), (4, 5) \\
\text{d} & : (-1, 7), (1, -11) \\
\text{e} & : (-2, -6), (2, -8) \\
\text{f} & : (0, 4), (3, 0)
\end{align*} \]

9 a Find the equation of the straight line which passes through the point (1, 6) and is:

\[ \begin{align*}
\text{i} & : \text{parallel to the line with equation } y = 2x + 3 \\
\text{ii} & : \text{perpendicular to the line with equation } y = 2x + 3
\end{align*} \]

b Find the equation of the straight line which passes through the point (2, 3) and is:

\[ \begin{align*}
\text{i} & : \text{parallel to the line with equation } 4x + 2y = 10 \\
\text{ii} & : \text{perpendicular to the line with equation } 4x + 2y = 10
\end{align*} \]

10 Find the equation of the line which passes through the point of intersection of the lines \( y = x \) and \( x + y = 6 \) and which is perpendicular to the line with equation \( 3x + 6y = 12 \).

11 The length of the line joining \( A(2, -1) \) and \( B(5, y) \) is 5 units. Find \( y \).
12 Find the equation of the line passing through the point \((-1, 3)\) which is:
   a parallel to the lines with equations below
   b perpendicular to the lines with equations below
   i \(2x + 5y - 10 = 0\)  
   ii \(4x + 5y + 3 = 0\)

13 Find the angle that the lines joining the given points make with the positive direction of the x-axis:
   a \((-4, 1), (4, 6)\)  
   b \((2, 3), (-4, 6)\)  
   c \((5, 1), (-1, -8)\)  
   d \((-4, 2), (2, -8)\)

14 Find the acute angle between the lines \(y = 2x + 4\) and \(y = -3x + 6\)

15 Given the points \(A(a, 3), B(-2, 1)\) and \(C(3, 2)\), find the possible value of \(a\) if the length of \(AB\) is twice the length of \(BC\).

16 Three points have coordinates \(A(1, 7), B(7, 5)\) and \(C(0, -2)\). Find:
   a the equation of the perpendicular bisector of \(AB\)
   b the point of intersection of this perpendicular bisector and \(BC\)

17 The point \((h, k)\) lies on the line \(y = x + 1\) and is 5 units from the point \((0, 2)\). Write down two equations connecting \(h\) and \(k\) and hence find the possible values of \(h\) and \(k\).

18 \(P\) and \(Q\) are the points of intersection of the line \(\frac{y}{2} + \frac{x}{3} = 1\) with the x and y axes respectively. The gradient of \(QR\) is \(\frac{1}{2}\), where \(R\) is the point with x-coordinate \(2a, a > 0\).
   a Find the y-coordinate of \(R\) in terms of \(a\).
   b Find the value of \(a\) if the gradient of \(PR\) is \(-2\).

19 The figure shows a triangle \(ABC\) with \(A(1, 1), B(-1, 4)\).
   The gradients of \(AB, AC\) and \(BC\) are \(-3m, 3m\) and \(m\) respectively.
   a Find the value of \(m\).
   b Find the coordinates of \(C\).
   c Show that \(AC = 2AB\).

20 In the rectangle \(ABCD, A\) and \(B\) are the points \((4, 2)\) and \((2, 8)\) respectively. Given that the equation of \(AC\) is \(y = x - 2\), find:
   a the equation of \(BC\)
   b the coordinates of \(C\)
   c the coordinates of \(D\)
   d the area of rectangle \(ABCD\).
21 *ABCD* is a parallelogram, lettered anticlockwise, such that *A* and *C* are the points \((-1, 5)\) and \((5, 1)\) respectively.

- **a** Find the coordinates of the midpoint of *AC*.
- **b** Given that *BD* is parallel to the line whose equation is \(y + 5x = 2\), find the equation of *BD*.
- **c** Given that *BC* is perpendicular to *AC*, find:
  - **i** the equation of *BC*
  - **ii** the coordinates of *B*
  - **iii** the coordinates of *D*.

### 2.4 Applications of linear functions

#### Example 4

There are two possible methods for paying gas bills:

- method A: a fixed charge of $25 per quarter + 50c per unit of gas used
- method B: a fixed charge of $50 per quarter + 25c per unit of gas used.

Determine the number of units which must be used before method B becomes cheaper than method A.

**Solution**

Let 
- \(C_1\) = charge in $ using method A
- \(C_2\) = charge in $ using method B

\[ x = \text{number of units of gas used.} \]

Now 
- \(C_1 = 25 + 0.5x\)
- \(C_2 = 50 + 0.25x\)

It can be seen from the graph that if the number of units exceeds 100, then method B is cheaper.

The solution could also be obtained by solving simultaneous linear equations:

\[
C_1 = C_2 \\
25 + 0.5x = 50 + 0.25x \\
0.25x = 25 \\
x = 100
\]

#### Exercise 2D

1. A car journey of 300 km lasts 4 h. Part of this journey is on a freeway at an average speed of 90 km/h. The rest is on country roads at an average speed of 70 km/h.

Let \(T\) be the time (in hours) spent on the freeway.

   **a** In terms of \(T\), state the number of hours travelling on country roads.
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2 A farmer measured the quantity of water in a storage tank 20 days after it was filled and found it contained 3000 litres. After a further 15 days it was measured again and found to have 1200 litres in it. Assume that the amount of water in the tank decreases at a constant rate.

a Find the relation between \( L \), the number of litres of water in the tank, and \( t \), the number of days after the tank was filled.

b How much water does the tank hold when it is full?

c Sketch the graph of \( L \) against \( t \) for a suitable domain.

d State this domain.

e How long does it take for the tank to empty?

f At what rate does the water leave the tank?

3 On a small island two rival taxi firms have the following fare structures:

- firm A: fixed charge of $1 plus 40 cents per kilometre
- firm B: 60 cents per kilometre, no fixed charge.

a Find expressions for \( C_A \), the charge of firm A in terms of \( n \), the number of kilometres travelled, and \( C_B \), the charge of firm B in terms of the number of kilometres travelled.

b Sketch the graph of the charges of each of the firms against the number of kilometres travelled on the one set of axes.

c Find the distance for which both firms charge the same amount.

d On a new set of axes sketch the graph of \( D = C_A - C_B \) against \( n \) and explain what this graph represents.

4 A boat leaves from \( O \) to sail to two islands. The boat arrives at a point \( A \) on Happy Island with coordinates (10, 22.5) (units are in kilometres).

a Find the equation of the line through points \( O \) and \( A \).

b Find the distance \( OA \) to the nearest metre.

The boat arrives at Sun Island at point \( B \).

The coordinates of point \( B \) are (23, 9).

c Find the equation of line \( AB \).

d A third island lies on the perpendicular bisector of line segment \( AB \). Its port is denoted by \( C \). It is known that the \( x \)-coordinate of \( C \) is 52.

Find the \( y \)-coordinate of the point \( C \).
5  

ABCD is a parallelogram with coordinates A(2, 2), B(1.5, 4) and C(6, 6).

a  Find the gradient of:
   i  line AB
   ii  line AD

b  Find the equation of:
   i  line BC
   ii  line CD

c  Find the equation of the diagonals AC and BD.

d  Find the coordinates of the point of intersection of the diagonals.

2.5  Review of matrix arithmetic

A matrix is a rectangular array of numbers.

- Two matrices A and B are equal when:
  - each has the same number of rows and the same number of columns
  - they have the same number or element at corresponding positions.

\[
\begin{bmatrix}
3 & 6 & 5 \\
-6 & 10 & 6 \\
12 & 1 & 0
\end{bmatrix}
= \begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
\]

implies \( a = 3, b = 6, c = 5, \) etc.

- The size or **dimension** of a matrix is described by specifying the number of rows (m) and the number of columns (n). The dimension is written \( m \times n \). The dimensions of the following matrices in order are:

\( 3 \times 2, \ 1 \times 4, \ 3 \times 3, \ 1 \times 1 \)

\[
\begin{bmatrix}
-1 & 2 \\
-3 & 4 \\
5 & 6
\end{bmatrix}
+ \begin{bmatrix}
2 & 1 & 5 & 6
\end{bmatrix}
= \begin{bmatrix}
\sqrt{2} & \pi & 3 \\
0 & 0 & 1 \\
\sqrt{2} & 0 & \pi
\end{bmatrix}
\]

- Addition will be defined for two matrices only when they have the same dimension. The sum is found by adding corresponding elements.

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
+ \begin{bmatrix}
e & f \\
g & h
\end{bmatrix}
= \begin{bmatrix}
a + e & b + f \\
c + g & d + h
\end{bmatrix}
\]

Subtraction is defined in a similar way.
If $A$ is an $m \times n$ matrix and $k$ is a real number, $kA$ is defined to be an $m \times n$ matrix whose elements are $k$ times the corresponding element of $A$.

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

If $A$ is an $m \times n$ matrix and $B$ is an $n \times r$ matrix, then the product $AB$ is the $m \times r$ matrix whose entries are determined as follows.

To find the entry in row $i$ and column $j$ of $AB$, single out row $i$ in matrix $A$ and column $j$ in matrix $B$. Multiply the corresponding entries from the row and column and then add up the resulting products.

The product $AB$ is defined only if the number of columns of $A$ is the same as the number of rows of $B$.

For $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$

$A$ is a $2 \times 3$ matrix and $B$ is a $3 \times 2$ matrix. Therefore $AB$ is a $2 \times 2$ matrix.

$$AB = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 \times 4 + 3 \times 1 + 4 \times 0 \\ 5 \times 4 + 6 \times 1 + 7 \times 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 26 \end{bmatrix}$$

The identity matrix for the family of $n \times n$ matrices is the matrix with ones in the ‘top left’ to ‘bottom right’ diagonal and zeros in all other positions. This is denoted by $I$.

If $A$ and $B$ are square matrices of the same dimension and $AB = BA = I$ then $A$ is said to be the inverse of $B$ and $B$ is said to be the inverse of $A$. The inverse of $A$ is denoted by $A^{-1}$.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} d & -b \\ -c & ad - bc \end{bmatrix}$

$$\det(A) = ad - bc$$ is the determinant of matrix $A$.

A square matrix is said to be regular if its inverse exists. Those square matrices which do not have an inverse are called singular matrices. The determinant of a singular matrix is 0.
Using the TI-Nspire

Entering matrices using the template

2-by-2 matrices are easiest entered using the 2-by-2 matrix template (or as shown. Notice that there is also a template for entering $m$-by-$n$ matrices.

To enter the matrix $A = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$, use the Nav Pad to move between the entries of the 2-by-2 matrix template and store (the matrix as $a$.

Define the matrix $B = \begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix}$ in a similar way.

Alternatively, enter the matrix by typing [3,6;6,7]. Semicolon (;) can be obtained by typing (as shown).
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Entering matrices directly

To enter matrix $A$ without using the template, enter the matrix row by row as $\begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ and store the matrix as $a$.

Addition, subtraction and multiplication by a scalar

Once $A$ and $B$ are defined as above, $A + B$, $A - B$ and $kA$ can easily be determined.

Matrix multiplication

Multiplication of $A = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$, and $B = \begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix}$.

The products $AB$ and $BA$ are shown.

Inverse and determinants

The operation of matrix inverse is obtained by raising the matrix to the power of $-1$.

The Determinant command is found in the Matrix and Vector menu and used as shown.
Using the Casio ClassPad

Matrix editor

To enter matrices, in Main press Keyboard, then in the 2D CALC screen tap one of the matrix entry buttons \([\text{[ }\]],[\text{[ }\]],[\text{[ }\]],[\text{[ }\]) to enter row, column or square matrices.

To add an extra column to a matrix tap \([\text{[ }\]],[\text{[ }\]],[\text{[ }\]) .
To add an extra row tap \([\text{[ }\]],[\text{[ }\]],[\text{[ }\]) .
To expand in both directions, tap \([\text{[ }\]],[\text{[ }\]],[\text{[ }\]) .
Enter the matrices \(A\) and \(B\) as shown using \([\text{[ }\]],[\text{[ }\]],[\text{[ }\]],[\text{[ }\]],[\text{[ }\]) to store them as variables in the 2D OPTN tab menu.

Addition, subtraction, multiplication by a scalar

Once \(A\) and \(B\) are defined as above, \(A + B\), \(A - B\) and \(kB\) can be determined.

Multiplication

Multiplication \(AB\) is shown. Verify for yourself, by entering \(BA\), that multiplication is not commutative.
Inverse and determinant

To find the inverse of a matrix enter \( A^{-1} \). To find the determinant of the matrix, tap Interactive – Matrix – Calculation – det and enter the matrix \( A \).

**Example 5**

For the matrix \( A = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix} \) find:

a. \( \det(A) \)

b. \( A^{-1} \)

c. \( X \) if \( AX = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix} \)

d. \( Y \) if \( YA = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix} \)

**Solution**

a. \( \det(A) = 3 \times 6 - 2 = 16 \)

b. \( A^{-1} = \frac{1}{16} \begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix} \)

c. \( AX = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix} \)

Multiply both sides (from the left) by \( A^{-1} \).

\[
A^{-1}AX = A^{-1} \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}
\]

\[
\therefore IX = X = \frac{1}{16} \begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 16 & 32 \\ 16 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}
\]
\[ d \quad YA = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix} \]

Multiply both sides (from the right) by \( A^{-1} \).

\[
YAA^{-1} = \frac{1}{16} \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}
\]

\[
\therefore \quad YI = Y = \frac{1}{16} \begin{bmatrix} 24 & 8 \\ 40 & -8 \end{bmatrix}
\]

\[
\therefore \quad Y = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{5}{2} & -1 \end{bmatrix}
\]

**Exercise 2E**

1. For the matrices \( A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix} \) find:
   - a. \( A + B \)
   - b. \( AB \)
   - c. \( BA \)
   - d. \( A - B \)
   - e. \( kA \)
   - f. \( 2A + 3B \)
   - g. \( A - 2B \)
   - h. \( \text{det}(A) \)
   - i. \( A^{-1} \)
   - j. \( \text{det}(B) \)
   - k. \( B^{-1} \)
   - l. \( \text{det}(AB) \)

2. Find the inverse of the following regular matrices (\( k \) is any non-zero real number):
   - a. \( \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \)
   - b. \( \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix} \)
   - c. \( \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \)

3. If \( A, B \) are the regular matrices
   \[
   A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \]
   find \( A^{-1}, B^{-1} \).
   Also find \( AB \) and hence find, if possible, \( (AB)^{-1} \).
   Also find from \( A^{-1}, B^{-1} \), the products \( A^{-1}B^{-1} \) and \( B^{-1}A^{-1} \). What do you notice?

4. Consider the matrix \( A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \).
   - a. Find \( A^{-1} \).
   - b. If \( AX = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix} \), find \( X \).
   - c. If \( YA = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix} \), find \( Y \).

5. If \( A = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix} \) and \( C = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix} \), find:
   - a. \( X \) such that \( AX + B = C \)
   - b. \( Y \) such that \( YA + B = C \)
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6 If $A$ is a $2 \times 2$ matrix, $a_{12} = a_{21} = 0$, $a_{11} \neq 0$, $a_{22} \neq 0$ then show that $A$ is regular and find $A^{-1}$.

7 For $A = \begin{bmatrix} 3 & 6 & 5 \\ -6 & 10 & 6 \\ 12 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 6 \\ 2 & 1 & 0 \end{bmatrix}$ find, using a calculator:
   a $A + B$
   b $AB$
   c $BA$
   d $3A$
   e $B^{-1}$
   f $A - B$
   g $A - 2B$
   h $A + 3B$
   i $(AB)^{-1}$
   j $B^{-1}A^{-1}$

8 For $A = \begin{bmatrix} 3 & 6 & 5 & -6 \\ 10 & 6 & 12 & 1 \\ 0 & 4 & 1 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 1 & 6 & 2 & 1 \\ 0 & 7 & 8 & 0 \end{bmatrix}$ find, using a calculator:
   a $AB$
   b $BA$
   c $3A$
   d $B^{-1}$
   e $A - B$
   f $A - 2B$

9 Solve each of the following matrix equations for $X$:
   a $AX = C$
   b $AX + D = C$
   where $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 6 \\ 2 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ and $D = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$

10 Solve each of the following matrix equations for $X$:
   a $AX = C$
   b $AX + D = C$
   where $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 1 & 6 & 2 & 1 \\ 0 & 7 & 8 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$ and $D = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$

2.6 Solving systems of linear simultaneous equations in two variables

Inverse matrices can be used to solve certain sets of simultaneous linear equations. Consider the equations:

\[3x - 2y = 5\]
\[5x - 3y = 9\]

This can be written as:

\[\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}\]
If \( A = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \) the determinant of \( A \) is \( 3(-3) - 5(-2) = 1 \) which is not zero and so \( A^{-1} \) exists.

\[
A^{-1} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}
\]

Multiplying the matrix equation \[
\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}
\]
on both sides on the left-hand side by \( A^{-1} \) and using the fact that \( A^{-1}A = I \) yields the following:

\[
A^{-1}\left( A \begin{bmatrix} x \\ y \end{bmatrix} \right) = A^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix}
\]

\[
\therefore I \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix}
\]

\[
\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ since } A^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\]

This is the solution to the simultaneous equations.

Check by substituting \( x = 3, y = 2 \) in the equations.

When dealing with simultaneous linear equations in two variables which represent straight lines that are parallel, then a singular matrix results.

For example the system

\[
\begin{align*}
    x + 2y &= 3 \\
    -2x - 4y &= 6
\end{align*}
\]

has associated matrix equation

\[
\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}
\]

Note that the determinant of \[
\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} = 1 \times (-4) - (-2 \times 2) = 0
\]

There is no unique solution to the system of equations.
Example 6

Solve the simultaneous equations

\[\begin{align*}
3x - 2y &= 6 \\
7x + 4y &= 7
\end{align*}\]

Solution

The matrix equation is

\[
\begin{bmatrix}
3 & -2 \\
7 & 4
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
6 \\
7
\end{bmatrix}
\]

Let \(A = \begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}\)

Then \(A^{-1} = \frac{1}{26} \begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}\)

and \(\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 38 \\ -21 \end{bmatrix}\)

Using the TI-Nspire

Enter the matrices as shown.

Both the 2-by-2 matrix template and the 2-by-1 matrix template can be found in the catalog (\(\text{Mat} \rightarrow \text{Create} \rightarrow \text{2x2 Matrix}\) or \(\text{Mat} \rightarrow \text{Create} \rightarrow \text{2x1 Matrix}\)).

**Note:** It is also possible to use

\[
solve \left( \begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right), x
\]

to find the values of \(x\) and \(y\).
Using the Casio ClassPad

Remember that a $2 \times 2$ matrix is said to be singular if its determinant is equal to 0. The matrix being singular can correspond to one of two situations:
1. There are infinitely many solutions.
2. There is no solution.

Example 7

Explain why the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 24$ have no solution.

Solution

The equations have no solution as they correspond to parallel lines and they are different lines. There is no point of intersection.

Each of the lines has gradient $\frac{-2}{3}$.

The matrix of the coefficients of $x$ and $y$ is $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ and the determinant of this matrix is 0. That is, the matrix is singular.

If, for a system of two linear equations with two variables $x$ and $y$, the $2 \times 2$ matrix of the coefficients of $x$ and $y$ is singular, then the system has either no solutions as discussed above or infinitely many solutions. This second case arises when the two equations represent the same line.
Example 8

The simultaneous equations $2x + 3y = 6$ and $4x + 6y = 12$ have infinitely many solutions. Describe these solutions through the use of a parameter.

Solution

Using the TI-Nspire

The parameter is a third variable. Note that the equations represent the same straight line. They both have gradient $-\frac{2}{3}$ and $y$-intercept 2.

Let $\lambda$ be this third variable.

In this case let $y = \lambda$. Then $x = \frac{-3(\lambda - 2)}{2}$ and the line can be described by

$$\left\{ \left( \frac{-3(\lambda - 2)}{2}, \lambda \right) : \lambda \in \mathbb{R} \right\}.$$

This may seem to make the situation unnecessarily complicated, but it is the solution given by the calculator, as shown opposite. The variable $c1$ takes the place of $\lambda$.

Using the Casio ClassPad

The ClassPad does not introduce a third variable.

The solution is given in terms of $y$.
Example 9

Consider the simultaneous linear equations
\[(m - 2)x + y = 2 \text{ and } mx + 2y = k\]
Find the values of \(m\) and \(k\) such that the system of equations has:

- a a unique solution
- b no solution
- c infinitely many solutions

Solution

Using a CAS calculator to find the solution
\[x = \frac{4 - k}{m - 4} \text{ and } y = \frac{k(m - 2) - 2m}{m - 4}\]

a The solution is unique if \(m \neq 4\) and \(k\) can be any real number.
b If \(m = 4\), the equations become
\[2x + y = 2 \text{ and } 4x + 2y = k\]
There is no solution if \(m = 4\) and \(k \neq 4\).
c If \(m = 4\) and \(k = 4\) there are infinitely many solutions as the equations are the same.

This method of expressing a solution generalises to the more complicated situation in three dimensions. This is also discussed in the next section.

Again it is noted that for a system of linear equations in two unknowns, the matrix of the coefficients of \(x\) and \(y\) being singular corresponds to either no solutions (parallel lines) or infinitely many solutions (the same line).

Exercise 2F

1 Solve each of the pairs of simultaneous linear equations using matrix methods:
   - a \(3x + 2y = 6\) and \(x - y = 7\)
   - b \(2x + 6y = 0\) and \(y - x = 2\)
   - c \(4x - 2y = 7\) and \(5x + 7y = 1\)
   - d \(2x - y = 6\) and \(4x - 7y = 5\)

2 Explain why the simultaneous equations \(2x + 3y = 6\) and \(4x + 6y = 10\) have no solution.

3 The simultaneous equations \(x - y = 6\) and \(2x - 2y = 12\) have infinitely many solutions. Describe these solutions through the use of a parameter.

4 Find the value of \(m\) for which the simultaneous equations
\[(m + 3)x + my = 12\]
\[(m - 1)x + (m - 3)y = 7\]

have no solution.
5  Find the value of $m$ for which the simultaneous equations
\[ 3x + my = 5 \text{ and } (m + 2)x + 5y = m \] have:
   a  an infinite number of solutions    b  no solutions

6  Consider the simultaneous equations
\[ mx + 2y = 8 \]
\[ 4x - (2 - m)y = 2m \]
   a  Find the values of $m$ for which there are:
      i  no solutions     ii  infinitely many solutions
   b  Solve the equations in terms of $m$, for suitable values of $m$.

7  a  Solve the simultaneous equations $2x - 3y = 4$ and $x + ky = 2$, where $k$ is a constant.
   b  Find the value of $k$ for which there is not a unique solution.

8  Find the values of $b$ and $c$ for which the equations $x + 5y = 4$ and $2x + by = c$ have:
   a  a unique solution    b  an infinite set of solutions    c  no solution

2.7  **Simultaneous linear equations with more than two variables**

Consider the general linear system of three equations in three unknowns:

\[
\begin{align*}
ax + by + cz &= d \\
ex + fy + gz &= h \\
kx + my + nz &= p
\end{align*}
\]

It can be written as a matrix equation:

\[
\begin{bmatrix}
a & b & c \\
e & f & g \\
k & m & n
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
d \\
h \\
p
\end{bmatrix}
\]

Let \( \mathbf{A} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & m & n \end{bmatrix} \), \( \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) and \( \mathbf{B} = \begin{bmatrix} d \\ h \\ p \end{bmatrix} \)

The equation is \( \mathbf{AX} = \mathbf{B} \)

We recall that for \( 3 \times 3 \) matrices \( \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) and \( \mathbf{D} = \mathbf{D} \mathbf{I} = \mathbf{I} \mathbf{D} \) for all \( 3 \times 3 \) matrices \( \mathbf{D} \).

If the inverse \( \mathbf{A}^{-1} \) exists (this is not always the case) the equation can be solved by multiplying \( \mathbf{AX} \) and \( \mathbf{B} \), on the left by \( \mathbf{A}^{-1} \), and \( \mathbf{A}^{-1}(\mathbf{AX}) = \mathbf{A}^{-1}\mathbf{B} \) and \( \mathbf{A}^{-1}(\mathbf{AX}) = (\mathbf{A}^{-1}A)\mathbf{X} = \mathbf{IX} = \mathbf{X} \), where \( \mathbf{I} \) is the identity matrix for \( 3 \times 3 \) matrices.
Hence \( X = A^{-1}B \), which is a formula for the solution of the system. Of course it depends on the inverse \( A^{-1} \) existing, but once \( A^{-1} \) is found then equations of the form \( AX = B \) can be solved for all possible \( 3 \times 1 \) matrices \( B \).

In this course you are not required to find the inverse of a \( 3 \times 3 \) matrix by hand but an understanding of matrix arithmetic is necessary.

**Example 10**

Consider the system of three equations in three unknowns:

\[
\begin{align*}
2x + y + z &= -1 \\
3y + 4z &= -7 \\
6x + z &= 8
\end{align*}
\]

Using matrix methods to solve the system of equations.

**Solution**

Enter \( 3 \times 3 \) matrix \( A \) and \( 3 \times 1 \) matrix \( B \) into the calculator.

\[
A = \begin{bmatrix}
2 & 1 & 1 \\
0 & 3 & 4 \\
6 & 0 & 1
\end{bmatrix}, \quad X = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
-1 \\
-7 \\
8
\end{bmatrix}
\]

The equations can be written as the matrix equation:

\[
AX = B
\]

Multiply both sides by \( A^{-1} \).

\[
A^{-1}AX = A^{-1}B \\
IX = A^{-1}B \\
X = A^{-1}B
\]

\[
X = \begin{bmatrix}
1 \\
-5 \\
2
\end{bmatrix}
\]

Hence \( x = 1 \), \( y = -5 \) and \( z = 2 \).

It should be noted that, just as for two equations in two unknowns, there is a geometric interpretation for three equations in three unknowns. There is only a unique solution if the equations represent three planes intersecting at a point.

A CAS calculator can be used to solve systems of three equations in the same way that was used for two simultaneous equations. The solution of equations without CAS is considered in Exercise 2G.
Example 11

Solve the linear simultaneous equations

\[ x - y + z = 6, \ 2x + z = 4, \ 3x + 2y - z = 6 \]

for \( x, y \) and \( z \).

Solution

Using the TI-Nspire

The \textit{simultaneous equations} template \((\text{App} \rightarrow \text{Edit})\) has been used here, but the result could also be obtained using \texttt{solve(x - y + z = 6 and 2x + z = 4 and 3x + 2y - z = 6, x, y, z)}.

![Using the TI-Nspire](image)

Using the Casio ClassPad

Press \((\text{Key})\) and select the \(2D\) menu (if necessary tap \(\leftarrow\)).

Tap \(\rightarrow\) to enter two simultaneous equations and again to add a third.

At the bottom right, enter the variables \( x, y \) and \( z \).

The solution is as shown.

![Using the Casio ClassPad](image)
As a linear equation in two variables defines a line, a linear equation in three variables defines a plane. The coordinate axes in three dimensions are drawn as shown. The point $P(2, 2, 4)$ is as marked.

An equation of the form $ax + by + cy = e$ defines a plane. For example the equation $x - y + z = 6$ corresponds to the graph shown here.

**Using the Casio ClassPad**

Select 3D Graph from the main menu and enter $z1 = 6 - x + y$. Tap to select the graph then tap $\text{ }$ to produce the graph. If labels and axes do not appear, tap $\text{ }$ then 3D format and select the settings shown on the right. The graph may be zoomed and viewed from various angles by dragging the stylus on the screen. A full-screen view is available by selecting the graph frame (bold border) and tapping $\text{ }$.

The solution of simultaneous linear equations in three variables can correspond to
- a point
- a line
- a plane

There also may be no solution.
The situations are as shown:

Diagram 1
Intersection at a point

Diagram 2
Intersection, a line

Diagram 3
No intersection

Diagram 4
No common intersection

Diagram 5
No common intersection

Example 12

The simultaneous equations

\[ x + 2y + 3z = 13, \quad -x - 3y + 2z = 2 \]
\[ -x - 4y + 7z = 17 \]

have infinitely many solutions. Describe these solutions through the use of a parameter.

Solution

The equations have no unique solution. For example, the point \((-9, 5, 4)\) satisfies all three equations but it is certainly not the only one. We use the CAS calculator to find the solution in terms of a fourth variable, \(\lambda\). The calculator uses the parameter \(c\) for the parameter \(\lambda\).

Using the TI-Nspire

Let \(z = \lambda\), then \(y = 5(\lambda - 3)\) and
\(x = 43 - 13\lambda\)

If \(\lambda = 4\), \(x = -9\), \(y = 5\) and \(z = 4\).

Note that as \(z\) increases by 1 then \(y\) increases by 5 and \(x\) decreases by 13. All of the points which satisfy the equations lie on a straight line. The situation is similar to that shown in diagram 2 above.
Using the Casio ClassPad

The solution returned is:
\[ x = -13z + 43, \quad y = 5(z - 3), \quad z = z \]
The ClassPad does not introduce a parameter.

Exercise 2G

1 Solve each of the following sets of simultaneous equations using matrix methods:

a \[ 2x + 3y - z = 12 \quad b \quad x + 2y + 3z = 13 \quad c \quad x + y = 5 \]
\[ 2y + z = 7 \quad -x - y + 2z = 2 \quad y + z = 7 \]
\[ 2y - z = 5 \quad -x + 3y + 4z = 26 \quad z + x = 12 \]
\[ d \quad x - y - z = 0, \quad 5x + 20z = 50, \quad 10y - 20z = 30 \]

2 Consider the simultaneous equations \[ x + 2y - 3z = 4 \] and \[ x + y + z = 6 \]

a Subtract the second equation from the first to find y in terms of z.

b Let \( z = \lambda \). Solve the equations to give the solution in terms of \( \lambda \).

3 Consider the simultaneous equations

\[ x + 2y + 3z = 13 \quad \cdots \quad (1) \]
\[ -x - 3y + 2z = 2 \quad \cdots \quad (2) \]
\[ -x - 4y + 7z = 17 \quad \cdots \quad (3) \]

a Add equation (2) to equation (1) and subtract equation (2) from equation (3).

b Comment on the equations obtained in a.

c Let \( z = \lambda \) and find y in terms of \( \lambda \).

d Substitute for \( z \) and \( y \) in terms of \( \lambda \) in equation (1) to find \( x \) in terms of \( \lambda \).

4 Solve each of the following pairs of simultaneous equations, giving your answer in terms of a parameter \( \lambda \). Use the technique introduced in Question 2.

a \[ x - y + z = 4 \quad b \quad 2x - y + z = 6 \quad c \quad 4x - 2y + z = 6 \]
\[ -x + y + z = 6 \quad x - z = 3 \quad x + y + z = 4 \]

5 The system of equations \[ x + y + z + w = 4, \quad x + 3y + 3z = 2, \quad x + y + 2z - w = 6 \] has infinitely many solutions. Describe this family of solutions and give the unique solution when \( w = 6 \).
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6 Find all solutions of the following sets of equations:
   a $3x - y + z = 4$ and $x + 2y - z = 2$ and $-x + y - z = -2$
   b $x - y - z = 0$ and $3y + 3z = -5$
   c $2x - y + z = 0$ and $y + 2z = 2$

7 a Find the inverse of the matrix
   $\begin{bmatrix}
   2 & a & -1 \\
   3 & 4 & -(a + 1) \\
   10 & 8 & a - 4
   \end{bmatrix}$
in terms of $a$.
   b For what values of $a$ does the inverse not exist?
   c Find the value of $a$ for which there are infinitely many solutions to the equations
   $$2x + ay - z = 0$$
   $$3x + 4y - (a + 1)z = 13$$
   $$10x + 8y + (a - 4)z = 26$$
   d For the value of $a$ found in c, solve the equations.

8 Find a value of $p$ for which the system of equations
   $$3x + 2y - z = 1$$
   $$x + y + z = 2$$
   $$px + 2y - z = 1$$
has more than one solution.
   (Hint: Find the inverse of the matrix of coefficients in terms of $p$.)
Solve the system of equations for this value of $p$. 
Chapter summary

- Gradient of a straight line joining two points \((x_1, y_1)\) and \((x_2, y_2)\) is \(m = \frac{y_2 - y_1}{x_2 - x_1}\).
- Different forms for the equation of a straight line:
  - \(y = mx + c\) where \(m\) is the gradient and \(c\) is the \(y\)-axis intercept
  - \(y - y_1 = m(x - x_1)\) where \(m\) is the gradient and \((x_1, y_1)\) is a point on the line
  - \(\frac{x}{a} + \frac{y}{b} = 1\) where \((a, 0)\) and \((0, b)\) are points on the line intersecting the axes.
- Tangent of the angle of slope \((\theta)\), \(\tan \theta = \frac{y_2 - y_1}{x_2 - x_1}\), where \(\theta\) is the angle the line makes with the positive direction of the \(x\)-axis.
- If two straight lines are perpendicular to each other, the product of their gradients is \(-1\), i.e. \(m_1m_2 = -1\).
- Distance between two points \((x_1, y_1)\) and \((x_2, y_2)\) = \(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\).
- Midpoint of a straight line joining \((x_1, y_1)\) and \((x_2, y_2)\) is the point \(\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\).
- The angle \(\alpha\) between intersecting straight lines is as shown: \(\alpha = \theta_2 - \theta_1\)

- A matrix is a rectangular array of numbers.
- Two matrices \(A\) and \(B\) are equal when:
  - each has the same number of rows and the same number of columns
  - they have the same number or element at corresponding positions.
- The size or dimension of a matrix is described by specifying the number of rows \((m)\) and the number of columns \((n)\). The dimension is written \(m \times n\).
- Addition will be defined for two matrices only when they have the same dimension. The sum is found by adding corresponding elements.

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}
\]

Subtraction is defined in a similar way.
- If \(A\) is an \(m \times n\) matrix and \(k\) is a real number, \(kA\) is defined to be an \(m \times n\) matrix whose elements are \(k\) times the corresponding element of \(A\).

\[
k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}
\]
If $A$ is an $m \times n$ matrix and $B$ is an $n \times r$ matrix, then the product $AB$ is the $m \times r$ matrix whose entries are determined as follows.

To find the entry in row $i$ and column $j$ of $AB$, single out row $i$ in matrix $A$ and column $j$ in matrix $B$. Multiply the corresponding entries from the row and column and then add up the resulting products.

The product $AB$ is defined only if the number of columns of $A$ is the same as the number of rows of $B$.

The identity matrix $I$ satisfies the property that $AI = IA = A$.

If $A$ and $B$ are square matrices of the same dimension and $AB = BA = I$ then $A$ is said to be the inverse of $B$ and $B$ is said to be the inverse of $A$. The inverse of $A$ is denoted $A^{-1}$.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} d & -b \\ ad-bc & \frac{ad-bc}{ad-bc} \end{bmatrix}$.

$\det(A) = ad - bc$ is the **determinant** of matrix $A$.

A square matrix is said to be **regular** if its inverse exists. Those square matrices that do not have an inverse are called **singular** matrices.

Simultaneous equations can be solved using inverse matrices, For example

$$ax + by = c$$
$$dx + ey = f$$

can be written as $\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1} \begin{bmatrix} c \\ f \end{bmatrix}$

**Multiple-choice questions**

1. A straight line has gradient $-\frac{1}{2}$ and passes through $(1, 4)$. The equation of the line is:
   A. $y = x + 4$
   B. $y = 2x + 2$
   C. $y = 2x + 4$
   D. $y = -\frac{1}{2}x + 4$
   E. $y = -\frac{1}{2}x + \frac{9}{2}$

2. The line $y = -2x + 4$ passes through a point $(a, 3)$. The value of $a$ is:
   A. $-\frac{1}{2}$
   B. $2$
   C. $-\frac{7}{2}$
   D. $-2$
   E. $\frac{1}{2}$

3. The gradient of a line that is perpendicular to the line shown could be:
   A. $1$
   B. $\frac{1}{2}$
   C. $-\frac{1}{2}$
   D. $2$
   E. $-2$
4 The coordinates of the midpoint of $AB$ where $A$ has coordinates $(1, 7)$ and $B$ has coordinates $(-3, 10)$ are:
   A $(-2, 3)$  B $(-1, 8)$  C $(-1, 8.5)$
   D $(-1, 3)$  E $(-2, 8.5)$

5 The solution of the two simultaneous equations $ax - 5by = 11$ and $4ax + 10by = 2$ for $x$ and $y$, in terms of $a$ and $b$, is:
   A $x = \frac{10}{a}, y = -\frac{21}{5b}$  B $x = \frac{4}{a}, y = -\frac{7}{5b}$
   C $x = \frac{13}{5a}, y = -\frac{42}{25b}$  D $x = \frac{13}{2a}, y = -\frac{9}{10b}$
   E $x = \frac{3}{a}, y = -\frac{14}{5b}$

6 The gradient of the line passing through $(3, -2)$ and $(-1, 10)$ is:
   A $-3$  B $-2$  C $-\frac{1}{3}$  D 4  E 3

7 If two lines $-2x + y - 3 = 0$ and $ax - 3y + 4 = 0$ are parallel, then $a$ equals:
   A 6  B 2  C $\frac{1}{3}$  D $\frac{2}{3}$  E $-6$

8 A straight line passes through $(-1, -2)$ and $(3, 10)$. The equation of the line is:
   A $y = 3x - 1$  B $y = 3x - 4$  C $y = 3x + 1$
   D $y = \frac{1}{3}x + 9$  E $y = 4x - 2$

9 The length of the line segment connecting $(1, 4)$ and $(5, -2)$ is:
   A 10  B $2\sqrt{13}$  C 12  D 50  E $2\sqrt{5}$

10 The function with graph as shown has the rule:
   A $f(x) = 3x - 3$  B $f(x) = -\frac{3}{4}x - 3$
   C $f(x) = \frac{3}{4}x - 3$  D $f(x) = \frac{4}{3}x - 3$
   E $f(x) = 4x - 4$

Short-answer questions (technology-free)

1 Solve the following linear equations:
   a $3x - 2 = 4x + 6$  b $\frac{x + 1}{2x - 1} = \frac{4}{3}$  c $\frac{3x}{5} - 7 = 11$
   d $\frac{2x + 1}{5} = \frac{x - 1}{2}$

2 Solve each of the following pairs of simultaneous linear equations:
   a $y = x + 4$  b $\frac{x}{4} - \frac{y}{3} = 2$
   $5y + 2x = 6$  c $y - x = 5$
3 Sketch the graph of the relations:
   a $3y + 2x = 5$
   b $x - y = 6$
   c $\frac{x}{2} + \frac{y}{3} = 1$

4 a Find the equation of the straight line which passes through $(1, 3)$ and has gradient $-2$.
   b Find the equation of the straight line which passes through $(1, 4)$ and $(3, 8)$.
   c Find the equation of the straight line which is perpendicular to the line with equation $y = -2x + 6$ and which passes through the point $(1, 1)$.
   d Find the equation of the straight line which is parallel to the line with equation $y = 6 - 2x$ and which passes through the point $(1, 1)$.

5 Find the distance between the points with coordinates $(-1, 6)$ and $(2, 4)$.

**Extended-response questions**

1 A firm manufacturing jackets finds that it is capable of producing 100 jackets per day, but it can only sell all of these if the charge to wholesalers is no more than $50 per jacket. On the other hand, at the current price of $75 per jacket, only 50 can be sold per day.
   Assume that the graph of price, $P$, against number sold per day $N$ is a straight line.
   a Sketch the graph of $P$ against $N$.
   b Find the equation of the straight line.
   c Use the equation to find:
      i the price at which 88 jackets per day could be sold
      ii the number of jackets that should be manufactured to sell at $60 each.

2 A new town was built 10 years ago to house the workers of a woollen mill established in a remote country area. Three years after the town was built, it had a population of 12,000 people. Business in the wool trade steadily grew, and eight years after the town was built the population had swelled to 19,240.
   a Assuming the population growth can be modelled by a linear relationship, find a suitable relation for the population, $p$, in terms of $t$, the number of years since the town was built.
   b Sketch the graph of $p$ against $t$, and interpret the $p$-axis intercept.
   c Find the current population of the town.
   d Calculate the average rate of growth of the town.

3 $ABCD$ is a quadrilateral with angle $ABC$ a right angle. $D$ lies on the perpendicular bisector of $AB$. The coordinates of $A$ and $B$ are $(7, 2)$ and $(2, 5)$ respectively. The equation of line $AD$ is $y = 4x - 26$.
   a Find the equation of the perpendicular bisector of line segment $AB$.
   b Find the coordinates of point $D$.
   c Find the gradient of line $BC$.
   d Find the value of the second coordinate $c$ of the point $C(8, c)$.
   e Find the area of quadrilateral $ABCD$. 
4 Triangle $ABC$ is isosceles with $BC = AC$. The coordinates of the vertices are $A(6, 1)$ and $B(2, 8)$.
   a Find the equation of the perpendicular bisector of $AB$.
   b If the $x$-coordinate of $C$ is 3.5, find the $y$-coordinate of $C$.
   c Find the length of $AB$.
   d Find the area of triangle $ABC$.

5 If $A = (-4, 6)$ and $B = (6, -7)$, find:
   a the coordinates of the midpoint of $AB$
   b the length of $AB$
   c the distance between $A$ and $B$
   d the equation of $AB$
   e the equation of the perpendicular bisector of $AB$
   f the coordinates of $P$, where $P \in AB$ and $AP : PB = 3 : 1$
   g the coordinates of $P$, where $P \in AB$ and $AP : AB = 3 : 1$

6 A chemical manufacturer has an order for 500 litres of a 25% acid solution (i.e. 25% by volume is acid). Solutions of 30% and 18% are available in stock.
   a How much acid is required to produce 500 litres of 25% acid solution?
   b The manufacturer wishes to make up the 500 litres from a mixture of 30% and 18% solutions.
      Let $x$ denote the amount of 30% solution required.
      Let $y$ denote the amount of 18% solution required.
      Use simultaneous equations in $x$ and $y$ to determine the amount of each solution required.

7 Bronwyn and Noel have a clothing warehouse in Summerville. They are supplied by three contractors, Brad, Flynn and Elizabeth. The matrix below shows the number of dresses, slacks and shirts a worker, for each of the contractors, can produce in a week. The number produced varies because of the different equipment used by the contractors.

\[
\begin{bmatrix}
\text{Brad} & \text{Flynn} & \text{Elizabeth} \\
\text{Dresses} & 5 & 6 & 10 \\
\text{Slacks} & 3 & 4 & 5 \\
\text{Shirts} & 2 & 6 & 5 \\
\end{bmatrix}
\]

The warehouse requires 310 dresses, 175 slacks and 175 shirts in a week. How many workers should each contractor employ to meet this requirement exactly?