CHAPTER

1

Functions and relations

Objectives

- To understand and use the notation of sets, including the symbols $\in$, $\subseteq$, $\cap$, $\cup$, $\emptyset$ and $\setminus$.
- To use the notation for sets of numbers.
- To understand the concept of relation.
- To understand the terms domain and range.
- To understand the concept of function.
- To understand the term one-to-one.
- To understand the terms implied domain, restriction of a function, hybrid function, and odd and even functions.
- To understand the modulus function.
- To understand and use sums and products of functions.
- To define composite functions.
- To understand and find inverse functions.
- To apply a knowledge of functions to solving problems.

In this chapter, notation that will be used throughout the book will be introduced. The language introduced in this chapter is necessary for expressing important mathematical ideas precisely. If you are working with a CAS calculator it is appropriate to work through the first sections of the appropriate Computer Algebra System Appendix.

1.1 Set notation

Set notation is used widely in mathematics and in this book it is employed where appropriate. This section summarises much of the set notation you will need.

A set is a collection of objects. The objects that are in the set are known as the elements or members of the set. If $x$ is an element of a set $A$ we write $x \in A$. This can also be read as ‘$x$ is a member of the set $A$’ or ‘$x$ belongs to $A$’ or ‘$x$ is in $A$’.

The notation $x \notin A$ means $x$ is not an element of $A$.

For example: $2 \notin \text{set of odd numbers}$.

A set $B$ is called a subset of a set $A$ if and only if $x \in B$ implies $x \in A$. 
To indicate that $B$ is a subset of $A$, we write $B \subseteq A$. This expression can also be read as ‘$B$ is contained in $A$’ or ‘$A$ contains $B$’.

The set of elements common to two sets $A$ and $B$ is called the intersection of $A$ and $B$ and is denoted by $A \cap B$. Thus $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

If the sets $A$ and $B$ have no elements in common, we say $A$ and $B$ are disjoint, and write $A \cap B = \emptyset$.

The set $\emptyset$ is called the empty set or null set.

The union of sets $A$ and $B$, written $A \cup B$, is the set of elements that are either in $A$ or in $B$. This does not exclude objects that are elements of both $A$ and $B$.

**Example 1**

$A = \{1, 2, 3, 7\}$; $B = \{3, 4, 5, 6, 7\}$

Find:

a. $A \cap B$

b. $A \cup B$

**Solution**

a. $A \cap B = \{3, 7\}$

b. $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

**Note:** In this example, $3 \in A$ and $5 \notin A$ and $\{2, 3\} \subseteq A$.

Finally, the set difference of two sets $A$ and $B$ is denoted $A \setminus B$, where:

$$A \setminus B = \{x : x \in A, x \notin B\}$$

e.g., for $A$ and $B$ in Example 1, $A \setminus B = \{1, 2\}$ and $B \setminus A = \{4, 5, 6\}$

There will be a further discussion of set notation in Chapter 14, which will provide the additional notation necessary for the study of probability.

### Sets of numbers

The elements of the set $\{1, 2, 3, 4, \ldots\}$ are called the natural numbers. The set of natural numbers will be denoted by $N$.

The elements of $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ are called integers. The set of integers will be denoted by $Z$.

The numbers of the form $\frac{p}{q}$ with $p$ and $q$ integers, $q \neq 0$, are called rational numbers. The rational numbers may be characterised by the property that each rational number may be written as a terminating or recurring decimal. The set of rational numbers will be denoted by $Q$.

The real numbers that are not rational numbers are called irrational (e.g., $\pi$ and $\sqrt{2}$).

The set of real numbers will be denoted by $R$.

It is clear that $N \subseteq Z \subseteq Q \subseteq R$ and this may be represented by the diagram:
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Note: \{x: 0 < x < 1\} is the set of all real numbers between 0 and 1.
\{x: x > 0, x \text{ rational}\} is the set of all positive rational numbers.
\{2n: n = 0, 1, 2, \ldots\} is the set of all even numbers.

Among the most important subsets of \( \mathbb{R} \) are the intervals. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that \( a \) and \( b \) are real numbers and that \( a < b \):

\[
\begin{align*}
(a, b) &= \{x: a < x < b\} & [a, b] &= \{x: a \leq x \leq b\} \\
(a, b] &= \{x: a < x \leq b\} & [a, b) &= \{x: a \leq x < b\} \\
(a, \infty) &= \{x: x > a\} & [a, \infty) &= \{x: x \geq a\} \\
(\infty, b) &= \{x: x < b\} & (-\infty, b] &= \{x: x \leq b\}
\end{align*}
\]

Intervals may be represented by diagrams, as shown in Example 2.

**Example 2**

Illustrate each of the following intervals of the real numbers on a number line:

a \([-2, 3]\)  
b \((-3, 4]\)  
c \((-\infty, 5]\)  
d \((-2, 4)\)  
e \((-3, \infty)\)

**Solution**

The ‘closed’ circle indicates that the number is included.
The ‘open’ circle indicates that the number is not included.

The following are also subsets of the real numbers for which there are special notations:

\[
\begin{align*}
R^+ &= \{x: x > 0\} \\
R^- &= \{x: x < 0\} \\
R\setminus\{0\} &= \text{the set of real numbers excluding 0.} \\
Z^+ &= \{x: x \in \mathbb{Z}, x > 0\}
\end{align*}
\]

The cartesian plane is denoted by \( \mathbb{R}^2 \) where \( \mathbb{R}^2 = \{(x, y): x \in \mathbb{R} \text{ and } y \in \mathbb{R}\} \)
Exercise 1A

1. For \( X = \{2, 3, 5, 7, 9, 11\} \), \( Y = \{7, 9, 15, 19, 23\} \) and \( Z = \{2, 7, 9, 15, 19\} \), find:
   a. \( X \cap Y \)
   b. \( X \cap Y \cap Z \)
   c. \( X \cup Y \)
   d. \( X \setminus Y \)
   e. \( Z \setminus Y \)
   f. \( X \cap Z \)
   g. \([-2, 8] \cap X \)
   h. \((-3, 8) \cap Y \)
   i. \((2, \infty) \cap Y \)
   j. \((3, \infty) \cup Y \)

2. For \( X = \{a, b, c, d, e\} \) and \( Y = \{a, e, i, o, u\} \), find:
   a. \( X \cap Y \)
   b. \( X \cup Y \)
   c. \( X \setminus Y \)
   d. \( Y \setminus X \)

3. For \( A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \), \( B = \{2, 4, 6, 8, 10\} \) and \( C = \{1, 3, 6, 9\} \), find:
   a. \( B \cap C \)
   b. \( B \setminus C \)
   c. \( A \setminus B \)
   d. \( (A \setminus B) \cup (A \setminus C) \)
   e. \( A \setminus (B \cap C) \)
   f. \((A \setminus B) \cap (A \setminus C) \)
   g. \( A \setminus (B \cup C) \)
   h. \( A \cap B \cap C \)

4. Use the appropriate interval notation, i.e. \([a, b]\), \((a, b)\) etc., to describe each of the following sets:
   a. \( \{x: -3 \leq x < 1\} \)
   b. \( \{x: -4 < x \leq 5\} \)
   c. \( \{y: -\sqrt{2} < y < 0\} \)
   d. \( \left\{ x: -\frac{1}{\sqrt{2}} < x < \sqrt{3} \right\} \)
   e. \( \{x: x < -3\} \)
   f. \( \mathbb{R}^+ \)
   g. \( \mathbb{R}^- \)
   h. \( \{x: x \geq -2\} \)

5. Describe each of the following subsets of the real number line using the notation \([a, b]\), \((a, b)\), etc.:
   a.
   \[
   \begin{array}{cccccccccc}
   & & & & & & & & & \\
   & & & -4 & -3 & -2 & 0 & 1 & 2 & 3 & 4 & 5 \\
   \end{array}
   \]
   b.
   \[
   \begin{array}{cccccccccc}
   & & & & & & & & & \\
   & & & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
   \end{array}
   \]
   c.
   \[
   \begin{array}{cccccccccc}
   & & & & & & & & & \\
   & & & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
   \end{array}
   \]
   d.
   \[
   \begin{array}{cccccccccc}
   & & & & & & & & & \\
   & & & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
   \end{array}
   \]

6. Illustrate each of the following intervals on a number line:
   a. \((-3, 2)\)
   b. \((-4, 3)\)
   c. \((-\infty, 3)\)
   d. \([-4, -1]\)
   e. \([-4, \infty)\)
   f. \([-2, 5)\)

7. For each of the following, use one number line on which to represent the sets:
   a. \([-3, 6], [2, 4], [-3, 6] \cap [2, 4]\)
   b. \([-3, 6], \mathbb{R} \setminus [-3, 6]\)
   c. \([-2, \infty), (-\infty, 6], [-2, \infty) \cap (-\infty, 6]\)
   d. \((-8, -2), \mathbb{R}^- \setminus (-8, -2)\)
1.2 Identifying and describing relations and functions

An ordered pair, denoted \((a, b)\), is a pair of elements \(a\) and \(b\) in which \(a\) is considered to be the first element and \(b\) the second. In this section, only ordered pairs of real numbers are considered.

Two ordered pairs \((a, b)\) and \((c, d)\) are equal if \(a = c\) and \(b = d\).

A relation is a set of ordered pairs. The following are examples of relations:

\[
S = \{(1, 1), (1, 2), (3, 4), (5, 6)\}
\]

\[
T = \{(-3, 5), (4, 12), (5, 12), (7, -6)\}
\]

Every relation determines two sets defined as follows:

- The **domain** of a relation \(S\) is the set of all first elements of the ordered pairs in \(S\).
- The **range** of a relation \(S\) is the set of all second elements of the ordered pairs in \(S\).

In the above examples:
- domain of \(S\) = \(\{1, 3, 5\}\); range of \(S\) = \(\{1, 2, 4, 6\}\)
- domain of \(T\) = \(\{-3, 4, 5, 7\}\); range of \(T\) = \(\{5, 12, -6\}\)

A relation may be defined by a rule which pairs the elements in its domain and range. Thus the set

\[
\{(x, y) : y = x + 1, x \in \{1, 2, 3, 4\}\}
\]

is the relation

\[
\{(1, 2), (2, 3), (3, 4), (4, 5)\}
\]

When the domain of a relation is not explicitly stated, it is understood to consist of all real numbers for which the defining rule has meaning. For example:

\[
S = \{(x, y) : y = x^2\}
\]

is assumed to have domain \(\mathbb{R}\) and

\[
T = \{(x, y) : y = \sqrt{x}\}
\]

is assumed to have domain \([0, \infty)\).

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**Example 3**

Sketch the graph of each of the following relations and state the domain and range of each.

- \(a\) \(\{(x, y) : y = x^2\}\)
- \(b\) \(\{(x, y) : y \leq x + 1\}\)
- \(c\) \(\{(-2, -1), (-1, -1), (-1, 1), (0, 1), (1, -1)\}\)
- \(d\) \(\{(x, y) : x^2 + y^2 = 1\}\)
- \(e\) \(\{(x, y) : 2x + 3y = 6, x \geq 0\}\)
- \(f\) \(\{(x, y) : y = 2x - 1, x \in [-1, 2]\}\)
Solution

**a**

Domain = \( R \); range = \( R^{+} \cup \{0\} \)

**b**

Domain = \( R \); range = \( R \)

**c**

Domain = \( \{-2, -1, 0, 1\} \);
range = \( \{-1, 1\} \)

**d**

Domain = \( \{x: -1 \leq x \leq 1\} \);
range = \( \{-1, 1\} \)

**e**

Domain = \( [0, \infty) \);
range = \( (-\infty, 2] \)

**f**

Domain = \( [-1, 2] \);
range = \( [-3, 3] \)

Sometimes the set notation is not used in the specification of a relation.
For the above example:

- **a** is written as \( y = x^2 \)
- **b** is written as \( y \leq x + 1 \)
- **e** is written as \( 2x + 3y = 6, x \geq 0 \)
A function is a relation such that no two ordered pairs of the relation have the same first element. For instance, in Example 3, a, e and f are functions but b, c and d are not.

Let S be a relation with domain D. A simple geometric test to determine if S is a function is as follows.

Consider the graph of S. If all vertical lines with equations \( x = a, a \in D \), cut the graph of S only once, then S is a function.

For example,

\[
x^2 + y^2 = 1 \text{ is not a function} \quad y = x^2 \text{ is a function}
\]

Functions are usually denoted by lower case letters such as \( f, g, h \).

The definition of a function tells us that for each \( x \) in the domain of \( f \) there is a unique element, \( y \), in the range such that \( (x, y) \in f \). The element \( y \) is called the image of \( x \) under \( f \) or the value of \( f \) at \( x \) and is denoted by \( f(x) \) (read ‘\( f \) of \( x \)’).

If \( (x, y) \in f \), then \( x \) is called a pre-image of \( y \).

This gives an alternative way of writing functions.

1 For the function \( \{(x, y); y = x^2\} \), write:

\[
f: R \to R, \quad f(x) = x^2
\]

2 For the function \( \{(x, y); y = 2x - 1, x \in [0, 4]\} \) write:

\[
f: [0, 4] \to R, \quad f(x) = 2x - 1
\]

3 For the function \( \{(x, y); y = \frac{1}{x}\} \), write:

\[
f: R \setminus \{0\} \to R, \quad f(x) = \frac{1}{x}
\]

If the domain is \( R \) we often just write the rule, for example in 1 \( f(x) = x^2 \).

Note that in using the notation \( f: X \to Y, X \) is the domain but \( Y \) is not necessarily the range.

It is a set that contains the range and is called the codomain. With this notation for functions the domain of \( f \) is written as \( \text{dom } f \) and range of \( f \) as \( \text{ran } f \).
Using the TI-Nspire

Function notation can be used with a CAS calculator.

Define \((b11)\) the function \(f(x) = 4x - 3\).
Type \(f(-3)\) followed by enter to evaluate \(f(-3)\).
Type \(f([1, 2, 3])\) followed by enter to evaluate \(f(1), f(2)\) and \(f(3)\).

Using the Casio ClassPad

Function notation can be used with a CAS calculator.

In \(\text{Main}\), tap \text{Interactive–Define} and enter the function name, variable and expression as shown.
See page 10 for a screen showing the Define window.
Enter \(f(-3)\) in the entry line and tap \(\text{Ex} \).
In the entry line, type \(f([1, 2, 3])\) to obtain the values of \(f(1), f(2)\) and \(f(3)\).

Example 4

If \(f(x) = 2x^2 + x\), find \(f(3), f(-2)\) and \(f(x - 1)\).

Solution

\[
\begin{align*}
f(3) &= 2(3)^2 + 3 = 21 \\
f(-2) &= 2(-2)^2 - 2 = 6 \\
f(x - 1) &= 2(x - 1)^2 + x - 1 \\
&= 2(x^2 - 2x + 1) + (x - 1) \\
&= 2x^2 - 3x + 1
\end{align*}
\]
Example 5

If $f(x) = 2x + 1$, find $f(-2)$ and $f\left(\frac{1}{a}\right)$, $a \neq 0$.

Solution

$$f(-2) = 2(-2) + 1 = -3$$

$$f\left(\frac{1}{a}\right) = 2\left(\frac{1}{a}\right) + 1 = \frac{2}{a} + 1$$

Using the TI-Nspire

Define ( keystroke 1 ) the function

$f(x) = 2x + 1$.

Type $f(-2)$ followed by enter to evaluate $f(-2)$.

Type $f\left(\frac{1}{a}\right)$ followed by enter to evaluate $f\left(\frac{1}{a}\right)$.

Using the Casio ClassPad

In tap Interactive–Define and enter

the function name $f$, variable $x$ and

expression $2x + 1$.

Now complete $f(-2)$ and $f\left(\frac{1}{a}\right)$.

Example 6

Consider the function defined by $f(x) = 2x - 4$ for all $x \in R$.

a Find the value of $f(2)$, $f(-1)$ and $f(t)$.

b For what values of $t$ is $f(t) = t$?

c For what values of $x$ is $f(x) \geq x$?

d Find the pre-image of 6.
**Solution**

a  \[ f(2) = 2(2) - 4 \]
\[ = 0 \]
\[ f(-1) = 2(-1) - 4 \]
\[ = -6 \]
\[ f(t) = 2t - 4 \]

b  \[ f(t) = t \]
\[ 2t - 4 = t \]
\[ t - 4 = 0 \]
\[ \therefore t = 4 \]

c  \[ f(x) \geq x \]
\[ 2x - 4 \geq x \]
\[ x - 4 \geq 0 \]
\[ \therefore x \geq 4 \]

d  \[ f(x) = 6 \]
\[ 2x - 4 = 6 \]
\[ x = 5 \]
\[ 5 \text{ is the pre-image of } 6. \]

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**Using the TI-Nspire**

Use **Define** (\(\boxed{\text{1} \ 1 \ 1}\)) and **Solve( )** (\(\boxed{\text{3} \ 1 \ 1}\)) as shown.

The symbol \(\geq\) can be found in the catalog (\(\boxed{\text{4} \ 4}\)), by typing \(\boxed{\text{m} \ >}\) or by typing \(>\) \(\boxed{\text{e}}\).

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**Using the Casio ClassPad**

Tap **Interactive–Define** and enter the function name, variable and expression as shown.

Enter and highlight \(f(x) = x\), tap **Interactive–Equation/inequality–solve** and ensure the variable is set as \(x\).

To enter the inequality, press **(app)\(m\)** and look in the **(app)\(\text{OPTN}\)** to find the \(\geq\) symbol.


**Restriction of a function**

Consider the following functions:

\[ f(x) = x^2, \ x \in \mathbb{R} \]

\[ g(x) = x^2, \ -1 \leq x \leq 1 \]

\[ h(x) = x^2, \ x \in \mathbb{R}^+ \cup \{0\} \]

The different letters, \( f, g \) and \( h \), used to name the functions, emphasise the fact that there are three different functions even though they each have the same rule. They are different because they are defined for different domains. We call \( g \) and \( h \) restrictions of \( f \) since their domains are subsets of the domain of \( f \).

**Example 7**

For each of the following, sketch the graph and state the range:

\( a \quad f: [-2, 4] \rightarrow \mathbb{R}, \ f(x) = 2x - 4 \)

\( b \quad g: (-1, 2] \rightarrow \mathbb{R}, \ g(x) = x^2 \)

**Solution**

\( a \)

\( b \)

Range = \([-8, 4]\)

Range = \([0, 4]\)
Exercise 1B

1 State the domain and range for the relations represented by each of the following graphs:

![Graphs](image)

2 Sketch the graph of each of the following relations and state the domain and range of each:

- \( \{(x, y) : y = x^2 + 1\} \)
- \( \{(x, y) : 3x + 12y = 24, x \geq 0\} \)
- \( \{(x, y) : y = 5 - x, x \in [0, 5]\} \)
- \( y = 3x - 2, x \in [-1, 2] \)
- \( y = \sqrt{2x} \)
- \( y = x^2 + 2, x \in [0, 4] \)
- \( y = 4 - x^2 \)

3 Which of the following relations are functions? State the domain and range for each.

- \( \{(-1, 1), (-1, 2), (1, 2), (3, 4), (2, 3)\} \)
- \( \{(-2, 0), (-1, -1), (0, 3), (1, 5), (2, -4)\} \)
- \( \{(-1, 1), (-1, 2), (-2, -2), (2, 4), (4, 6)\} \)
- \( \{(-1, 4), (0, 4), (1, 4), (2, 4), (3, 4)\} \)
- \( \{(x, 4) : x \in R\} \)
- \( y = -2x + 4 \)
- \( \{x, y) : x^2 + y^2 = 16\} \)

4 Consider the function \( g(x) = 3x^2 - 2 \).

- Find \( g(-2) \), \( g(4) \).
- State the range of \( g \).

5 Let \( f(x) = 2x^2 + 4x \) and \( g(x) = 2x^3 + 2x - 6 \).

- Evaluate \( f(-1) \), \( f(2) \) and \( f(-3) \).
- Evaluate \( g(-1) \), \( g(2) \) and \( g(3) \).
- Express the following in terms of \( x \):
  - \( f(-2x) \)
  - \( f(x - 2) \)
  - \( g(-2x) \)
  - \( g(x + 2) \)
  - \( g(x^2) \)
6 Consider the function \( f(x) = 2x - 3 \). Find:
   a: the image of 3
   b: the pre-image of 11
   c: \( \{ x : f(x) = 4x \} \)

7 Consider the functions \( g(x) = 6x + 7 \) and \( h(x) = 3x - 2 \). Find:
   a: \( \{ x : g(x) = h(x) \} \)
   b: \( \{ x : g(x) > h(x) \} \)
   c: \( \{ x : h(x) = 0 \} \)

8 Rewrite each of the following using the \( f : X \rightarrow Y \) notation:
   a: \( \{ (x, y) : y = 2x + 3 \} \)
   b: \( \{ (x, y) : 3y + 4x = 12 \} \)
   c: \( \{ (x, y) : y = 2x - 3, x \geq 0 \} \)
   d: \( y = x^2 - 9, x \in \mathbb{R} \)
   e: \( y = 5x - 3, 0 \leq x \leq 2 \)

9 Sketch the graphs of each of the following and state the range of each:
   a: \( y = x + 1, x \in [2, \infty) \)
   b: \( y = -x + 1, x \in [2, \infty) \)
   c: \( y = 2x + 1, x \in [-4, \infty) \)
   d: \( y = 3x + 2, x \in (-\infty, 3) \)
   e: \( y = x + 1, x \in (-\infty, 3] \)
   f: \( y = 3x - 1, x \in [-2, 6] \)
   g: \( y = -3x - 1, x \in [-5, -1] \)
   h: \( y = 5x - 1, x \in (-2, 4) \)

10 For \( f(x) = 2x^2 - 6x + 1 \) and \( g(x) = 3 - 2x \):
   a: Evaluate \( f(2), f(-3), f(-2) \).
   b: Evaluate \( g(-2), g(1) \) and \( g(-3) \).
   c: Express the following in terms of \( a \):
      i: \( f(a) \)
      ii: \( f(a + 2) \)
      iii: \( g(-a) \)
      iv: \( g(2a) \)
      v: \( f(5 - a) \)
      vi: \( f(2a) \)
      vii: \( g(a) + f(a) \)
      viii: \( g(a) - f(a) \)

11 For \( f(x) = 3x^2 + x - 2 \), find:
   a: \( \{ x : f(x) = 0 \} \)
   b: \( \{ x : f(x) = x \} \)
   c: \( \{ x : f(x) = -2 \} \)
   d: \( \{ x : f(x) > 0 \} \)
   e: \( \{ x : f(x) > x \} \)
   f: \( \{ x : f(x) \leq -2 \} \)

12 For \( f(x) = x^2 + x \), find:
   a: \( f(-2) \)
   b: \( f(2) \)
   c: \( f(-a) \) in terms of \( a \)
   d: \( f(a) + f(-a) \) in terms of \( a \)
   e: \( f(a) - f(-a) \) in terms of \( a \)
   f: \( f(a^2) \) in terms of \( a \)

13 For \( g(x) = 3x - 2 \), find:
   a: \( \{ x : g(x) = 4 \} \)
   b: \( \{ x : g(x) > 4 \} \)
   c: \( \{ x : g(x) = a \} \)
   d: \( \{ x : g(-x) = 6 \} \)
   e: \( \{ x : g(2x) = 4 \} \)
   f: \( \{ x : \frac{1}{g(x)} = 6 \} , g(x) \neq 0 \)

14 Find the value of \( k \) for each of the following if \( f(3) = 3 \), where:
   a: \( f(x) = kx - 1 \)
   b: \( f(x) = x^2 - k \)
   c: \( f(x) = x^2 + kx + 1 \)
   d: \( f(x) = \frac{k}{x} \)
   e: \( f(x) = kx^2 \)
   f: \( f(x) = 1 - kx^2 \)

15 Find the values of \( x \) for which the given functions have the given value:
   a: \( f(x) = 5x - 4, f(x) = 2 \)
   b: \( f(x) = \frac{1}{x}, f(x) = 5 \)
   c: \( f(x) = \frac{1}{x^2}, f(x) = 9 \)
   d: \( f(x) = x + \frac{1}{x}, f(x) = 2 \)
   e: \( f(x) = (x + 1)(x - 2), f(x) = 0 \)
1.3 Types of functions and maximal domains

One-to-one and many-to-one functions

A function $f$ is said to be **one-to-one** if for $a, b \in \text{dom} \, f$, $a \neq b$, then $f(a) \neq f(b)$. In other words $f$ is called one-to-one if every image under $f$ has a unique pre-image.

The function $f(x) = 2x + 1$ is a one-to-one function. The function $f(x) = x^2$ is not a one-to-one function as, for example, $f(-3) = 9$ and $f(3) = 9$; i.e., 9 does not have a unique pre-image.

The function $f(x) = 5$ is not a one-to-one function as there are infinitely many pre-images of 5.

The function $f(x) = x^3$ is a one-to-one function.

A geometric test for a function to be one-to-one is as follows.

If for any $a \in \text{ran} \, f$ the horizontal line, $y = a$, crosses the graph of $f$ at only one point, the function is one-to-one.

A function that is not one-to-one is **many-to-one**.
Implied domains (maximal domains)

If the domain is unspecified, then the domain is the largest subset of \( \mathbb{R} \) for which the rule is defined. When the domain is not explicitly stated, it is implied by the rule. Thus for the function, \( f(x) = \sqrt{x} \) the implied domain (maximal domain) is \([0, \infty)\). We write:

\[
f: [0, \infty) \to \mathbb{R}, \ f(x) = \sqrt{x}
\]

**Example 8**

Find the implied domain of the functions with the following rules:

\( a \quad f(x) = \frac{2}{2x - 3} \)

\( b \quad g(x) = \sqrt{5 - x} \)

\( c \quad h(x) = \sqrt{x - 5} + \sqrt{8 - x} \)

\( d \quad f(x) = \sqrt{x^2 - 7x + 12} \)

**Solution**

\( a \quad f(x) \) is not defined when \( 2x - 3 = 0 \), i.e. when \( x = \frac{3}{2} \).

Thus the implied domain is \( \mathbb{R} \setminus \left\{ \frac{3}{2} \right\} \).

\( b \quad g(x) \) is defined when \( 5 - x \geq 0 \), i.e. when \( x \leq 5 \).

Thus the implied domain is \( (-\infty, 5] \).

\( c \quad h(x) \) is defined when \( x - 5 \geq 0 \) and \( 8 - x \geq 0 \), i.e. when \( x \geq 5 \) and \( x \leq 8 \).

Thus the implied domain is \( [5, 8] \).

\( d \quad f(x) \) is defined when \( x^2 - 7x + 12 \geq 0 \).

\[
x^2 - 7x + 12 \geq 0
\]

is equivalent to \( (x - 3)(x - 4) \geq 0 \).

Therefore, \( x \geq 4 \) or \( x \leq 3 \).

Thus the implied domain is \( (-\infty, 3] \cup [4, \infty) \).

**Hybrid functions**

**Example 9**

Sketch the graph of the function \( f \) given by:

\[
f(x) = \begin{cases} 
-x - 1 & \text{for } x < 0 \\
2x - 1 & \text{for } 0 \leq x \leq 1 \\
\frac{1}{2}x^2 + \frac{1}{2} & \text{for } x \geq 1
\end{cases}
\]
Essential Mathematical Methods 3 & 4 CAS

Solution

Functions like this, which have different rules for different subsets of the domain, are called hybrid functions.

Odd and even functions

An odd function has the property that $f(-x) = -f(x)$.

For example, $f(x) = x^3 - x$ is an odd function

since $f(-x) = (-x)^3 - (-x)$
\[= -x^3 + x\]
\[= -f(x)\]

An even function has the property that $f(-x) = f(x)$.

For example, $f(x) = x^2 - 1$ is an even function

since $f(-x) = (-x)^2 - 1$
\[= x^2 - 1\]
\[= f(x)\]

The graphs of even functions are symmetrical about the $y$-axis.

The properties of odd and even functions often facilitate the sketching of graphs.

Exercise 1C

State which of the following functions are one-to-one:

1. State which of the following functions are one-to-one:

   a. $\{(2, 3), (3, 4), (5, 4), (4, 6)\}$
   b. $\{(1, 2), (2, 3), (3, 4), (4, 6)\}$
   c. $\{(x, y): y = x^2 + 2\}$
   d. $\{(x, y): y = 2x + 4\}$
   e. $f(x) = 2 - x^2$
   f. $y = x^2, x \geq 1$
2 The following are graphs of relations.
   a State which are the graphs of a function.
   b State which are the graphs of a one-to-one function.

3 The graph of the relation \( \{(x, y) : y^2 = x + 2, x \geq -2\}\) is shown. From this relation, form two functions and specify the range of each.
4 a Draw the graph of \( g: R \rightarrow R, \ g(x) = x^2 + 2. \)

b By restricting the domain of \( g \), form two one-to-one functions that have the same rule as \( g \).

5 State the largest possible domain and range for the functions defined by the rule:

a \( y = 4 - x \) \quad b \( y = \sqrt{x} \) \quad c \( y = x^2 - 2 \) \quad d \( y = \sqrt{16 - x^2} \)

e \( y = \frac{1}{x} \) \quad f \( y = 4 - 3x^2 \) \quad g \( y = \sqrt{x - 3} \)

6 Each of the following is the rule of a function. In each case write down the implied domain and the range.

a \( y = 3x + 2 \) \quad b \( y = x^2 - 2 \) \quad c \( f(x) = \sqrt{9 - x^2} \) \quad d \( g(x) = \frac{1}{x - 1} \)

7 Find the implied domain for each of the following rules:

a \( f(x) = \frac{1}{\sqrt{x - 3}} \) \quad b \( f(x) = \sqrt{x^2 - 3} \) \quad c \( g(x) = \sqrt{x^2 + 3} \)

d \( h(x) = \sqrt{x - 4 + \sqrt{11 - x}} \) \quad e \( f(x) = \frac{x^2 - 1}{x + 1} \)

f \( h(x) = \sqrt{x^2 - x - 2} \) \quad g \( f(x) = \frac{1}{(x + 1)(x - 2)} \) \quad h \( h(x) = \frac{\sqrt{x - 1}}{\sqrt{x + 2}} \)

i \( f(x) = \sqrt{x - 3x^2} \) \quad j \( h(x) = \sqrt{25 - x^2} \) \quad k \( f(x) = \sqrt{x - 3 + \sqrt{12 - x}} \)

8 Which of the following functions are odd, even or neither?

a \( f(x) = x^4 \) \quad b \( f(x) = x^5 \) \quad c \( f(x) = x^4 - 3x \)

d \( f(x) = x^4 - 3x^2 \) \quad e \( f(x) = x^5 - 2x^3 \) \quad f \( f(x) = x^4 - 2x^5 \)

9 a Sketch the graph of the function:

\[
f(x) = \begin{cases} 
-2x - 2, & x < 0 \\
x - 2, & 0 \leq x < 2 \\
3x - 6, & x \geq 2
\end{cases}
\]

b What is the range of \( f \)?

10 State the domain and range of the function for which the graph is shown.
11 State the domain and range of the function for which the graph is shown.

12 a Sketch the graph of the function with rule:

\[ f(x) = \begin{cases} 
 2x + 6 & 0 < x \leq 2 \\
 -x + 5 & -4 \leq x \leq 0 \\
 -4 & x < -4 
\end{cases} \]

b State the domain and range of the function.

13 a Sketch the graph of the function with rule:

\[ g(x) = \begin{cases} 
 x^2 + 5 & x > 0 \\
 5 - x & -3 \leq x \leq 0 \\
 8 & x < -3 
\end{cases} \]

b State the range of the function.

14 Given that

\[ f(x) = \begin{cases} 
 \frac{1}{x}, & x > 3 \\
 \frac{x}{2x}, & x \leq 3 
\end{cases} \]

find:

a \( f(-4) \)  \hspace{1cm} b \( f(0) \)  \hspace{1cm} c \( f(4) \)  \hspace{1cm} d \( f(a + 3) \) in terms of \( a \)

f \( f(2a) \) in terms of \( a \)  \hspace{1cm} \( f(a - 3) \) in terms of \( a \)

15 Given that

\[ f(x) = \begin{cases} 
 \sqrt{x - 1}, & x \geq 1 \\
 4, & x < 1 
\end{cases} \]

find:

a \( f(0) \)  \hspace{1cm} b \( f(3) \)  \hspace{1cm} c \( f(8) \)

d \( f(a + 1) \) in terms of \( a \)  \hspace{1cm} \( f(a - 1) \) in terms of \( a \)

16 Sketch the graph of the function:

\[ g(x) = \begin{cases} 
 -x - 2, & x < -1 \\
 \frac{x - 1}{2}, & -1 \leq x < 1 \\
 3x - 3, & x \geq 1 
\end{cases} \]
17 Specify the function illustrated by the graph.

1.4 The modulus function

The modulus or absolute value of a real number \( x \) is denoted by \( |x| \) and is defined by:

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases}
\]

It may also be defined as \( |x| = \sqrt{x^2} \).

For example, \( |5| = 5 \) and \( |-5| = 5 \).

The function \( |x| \) has the following properties:

- \( |ab| = |a||b| \)
- \( \frac{a}{b} = \frac{|a|}{|b|} \)
- \( |a + b| \leq |a| + |b| \). If \( a \) and \( b \) are both non-negative or both non-positive, then equality holds.
- If \( a \geq 0 \), \( |x| \leq a \) is equivalent to \( -a \leq x \leq a \).
- If \( a \geq 0 \), \( |x - k| \leq a \) is equivalent to \( k - a \leq x \leq k + a \).

Example 10

Evaluate each of the following:

\begin{align*}
\text{a i} & \quad | -3 \times 2 | & \quad \text{ii} & \quad | -3 | \times | 2 | \\
\text{b i} & \quad \frac{-4}{2} & \quad \text{ii} & \quad \frac{|-4|}{|2|} \\
\text{c i} & \quad | -6 + 2 | & \quad \text{ii} & \quad |-6| + |2| \\
\end{align*}

Solution

\begin{align*}
\text{a i} & \quad | -3 \times 2 | = |-6| = 6 & \quad \text{ii} & \quad |-3| \times |2| = 3 \times 2 = 6 \\
\text{Note:} & \quad | -3 \times 2 | = |-3| \times |2| \\
\text{b i} & \quad \left| \frac{-4}{2} \right| = |-2| = 2 & \quad \text{ii} & \quad \frac{|-4|}{|2|} \quad \frac{4}{2} = 2 \\
\text{Note:} & \quad \left| \frac{-4}{2} \right| = \frac{|-4|}{|2|} \\
\text{c i} & \quad | -6 + 2 | = |-4| = 4 & \quad \text{ii} & \quad |-6| + |2| = 6 + 2 = 8 \\
\text{Note:} & \quad | -6 + 2 | \neq |-6| + |2| 
\end{align*}
Consider two points \( A \) and \( B \) on a number line:

\[
\begin{array}{c|cc}
& a & b \\
O & A & B \\
\end{array}
\]

On a number line the distance between points \( A \) and \( B \) is \(|a - b| = |b - a|\). Thus \(|x - 2| \leq 3\) can be read as ‘on the number line, the distance of \( x \) from 2 is less than or equal to 3’, and \(|x| \leq 3\) can be read as ‘on the number line, the distance of \( x \) from the origin is less than or equal to 3’. Note that \(|x| \leq 3\) is equivalent to \(-3 \leq x \leq 3\) or \(x \in [-3, 3]\).

The graph of the function \( f: \mathbb{R} \rightarrow \mathbb{R} \),

\[ f(x) = |x| \]

is as shown here.

Note that \(|x| = |-x|\), i.e. \(|x|\) is an even function.

**Example 11**

Illustrate each of the following sets on a number line and represent the sets using interval notation.

\begin{align*}
\text{a} & \quad \{x : |x| < 4\} \\
\text{b} & \quad \{x : |x| \geq 4\} \\
\text{c} & \quad \{x : |x - 1| \leq 4\}
\end{align*}

**Solution**

\begin{align*}
\text{a} & \quad (-4, 4) \\
\text{b} & \quad (-\infty, -4] \cup [4, \infty) \\
\text{c} & \quad [-3, 5]
\end{align*}

**Example 12**

Sketch the graphs of each of the following functions and state the range of each of the functions:

\begin{align*}
\text{a} & \quad f(x) = |x - 3| + 1 \\
\text{b} & \quad f(x) = -|x - 3| + 1
\end{align*}
Solution

First, note that \(|a - b| = a - b\) if \(a \geq b\) and \(|a - b| = b - a\) if \(b \geq a\).

\[ a \quad f(x) = |x - 3| + 1 = \begin{cases} x - 3 + 1 & \text{if } x \geq 3 \\ 3 - x + 1 & \text{if } x < 3 \end{cases} \]

\[ = \begin{cases} x - 2 & \text{if } x \geq 3 \\ 4 - x & \text{if } x < 3 \end{cases} \]

\[ b \quad f(x) = -|x - 3| + 1 = \begin{cases} -(x - 3) + 1 & \text{if } x \geq 3 \\ -(3 - x) + 1 & \text{if } x < 3 \end{cases} \]

\[ = \begin{cases} -x + 4 & \text{if } x \geq 3 \\ -2 + x & \text{if } x < 3 \end{cases} \]

Using the TI-Nspire

Complete as follows:

**Define** \( f(x) = \text{abs}(x - 3) + 1 \)

The absolute value function can be obtained by typing \( \text{abs} \), found as a command in the catalog (\(1\)A) or found as a template from the catalog (\(1\)B).

Open a **Graphs & Geometry** application (\(1\)A) and let \( f'(x) = f(x) \). Press **enter** to obtain the graph.

Note that the expression \( \text{abs} (x - 3) + 1 \) could have been entered directly for \( f'(x) \), but this way gives you greater flexibility to use the function in other ways if required.
Using the Casio ClassPad

Tap **Interactive-Define** and enter the function name, variable and function as shown.

To enter the absolute value, press **Keyboard** and look in the **Math** to find the \(|x|\) symbol.

In **Define**, enter \( f(x) \) into \( y_1 \), tick the box to select and tap **Graph** to create the graph.

Note that the expression could be directly entered in the \( 'y_1=' \) line but this gives you greater flexibility to use the function in other ways if required.

---

**Exercise 1D**

1. Evaluate each of the following:
   
   a. \(-5\) + 3
   b. \(-5\) + \(-3\)
   c. \(-5\) - \(-3\)
   d. \(-5\) - \(-3\) - 4
   e. \(-5\) - \(-3\) - \(-4\)
   f. \(-5\) + \(-3\) - \(-4\)

2. On a number line, illustrate each of the following sets and represent the sets using interval notation:
   
   a. \{x: |x| < 3\}
   b. \{x: |x| \(\geq\) 5\}
   c. \{x: |x - 2| \(\leq\) 1\}
   d. \{x: |x - 2| < 3\}
   e. \{x: |x + 3| \(\geq\) 5\}
   f. \{x: |x + 2| \(\leq\) 1\}

3. Sketch the graphs of each of the following functions and state the range of each of the functions:
   
   a. \( f(x) = |x - 4| + 1 \)
   b. \( f(x) = -|x + 3| + 2 \)
   c. \( f(x) = |x + 4| - 1 \)
   d. \( f(x) = 2 - |x - 1| \)
1.5 Sums and products of functions

The domain of \( f \) is denoted by \( \text{dom } f \) and the domain of \( g \) by \( \text{dom } g \). Let \( f \) and \( g \) be functions such that \( \text{dom } f \cap \text{dom } g \neq \emptyset \). The sum, \( f + g \), and the product, \( fg \), as functions on \( \text{dom } f \cap \text{dom } g \) are defined by:

1. \( (f + g)(x) = f(x) + g(x) \)
2. \( (fg)(x) = f(x)g(x) \)

The domain of both \( f + g \) and \( fg \) is the intersection of the domains of \( f \) and \( g \), i.e. the values of \( x \) for which both \( f \) and \( g \) are defined.

Graphing sums of functions will be discussed in Section 3.9.

Example 13

If \( f(x) = \sqrt{x - 2} \) for all \( x \geq 2 \) and \( g(x) = \sqrt{4 - x} \) for all \( x \leq 4 \), find:

\( f + g \quad (f + g)(3) \quad fg \quad (fg)(3) \)

Solution

\[ \begin{align*}
\text{a} & \quad \text{dom } f \cap \text{dom } g = [2, 4] \\
(f + g)(x) &= f(x) + g(x) \\
&= \sqrt{x - 2} + \sqrt{4 - x} \\
\text{dom } (f + g) &= [2, 4] \\
\text{b} & \quad (f + g)(3) = \sqrt{3 - 2} + \sqrt{4 - 3} = 2 \\
\text{c} & \quad (fg)(x) = f(x)g(x) \\
&= \sqrt{(x - 2)(4 - x)} \\
\text{d} & \quad (fg)(3) = \sqrt{(3 - 2)(4 - 3)} = 1 \\
\end{align*} \]

Exercise 1E

1. For each of the following, find \( (f + g)(x) \) and \( (fg)(x) \) and state the domain for both \( f + g \) and \( fg \):

\( \text{a} \quad f(x) = 3x, \quad g(x) = x + 2 \)
\( \text{b} \quad f(x) = 1 - x^2 \) for all \( x \in [-2, 2] \) and \( g(x) = x^2 \) for all \( x \in \mathbb{R}^+ \)
\( \text{c} \quad f(x) = \sqrt{x} \) and \( g(x) = \frac{1}{\sqrt{x}} \) for \( x \in [1, \infty) \)
\( \text{d} \quad f(x) = x^2, x \geq 0 \) and \( g(x) = \sqrt{4 - x}, 0 \leq x \leq 4 \)

2. Functions \( f, g, h, \) and \( k \) are defined by:

\( \text{i} \quad f(x) = x^2 + 1, x \in \mathbb{R} \quad \text{ii} \quad g(x) = x, x \in \mathbb{R} \)
\( \text{iii} \quad h(x) = \frac{1}{x^2}, x \neq 0 \quad \text{iv} \quad k(x) = \frac{1}{x}, x \neq 0 \)

\( \text{a} \) State which of the above functions are odd and which are even.
\( \text{b} \) Form the functions of \( f + h, f h, g + k, g k, f + g, f g \), stating which are odd and which are even.
1.6 Composite functions

A function may be considered to be similar to a machine for which the input (domain) is processed to produce an output (range).

For example, the following diagram represents an ‘f-machine’ where \( f(x) = 3x + 2 \)

\[
\begin{align*}
\text{INPUT} & \quad f(3) = 3 \times 3 + 2 = 11 \\
\text{OUTPUT} & \\
\end{align*}
\]

An alternative diagram is:

Input
\( 3 \)
\( f \)
\( 11 \)

Output
\( f(3) = 3 \times 3 + 2 = 11 \)
\( g(11) = 11^2 = 121 \)

With many processes, more than one machine operation is required to produce an output. Suppose an output is the result of one function being applied after another, e.g., \( f(x) = 3x + 2 \) followed by \( g(x) = x^2 \)

This is illustrated diagrammatically on the right.

A new function \( h \) is formed.

The rule for \( h \) is \( h(x) = (3x + 2)^2 \)

The diagram shows \( f(3) = 11 \) and then \( g(11) = 121 \).

This may be written:

\[
h(3) = g(f(3)) = g(11) = 121
\]

Similarly, \( h(-2) = g(f(-2)) = g(-4) = 16 \)

\( h \) is said to be the composition of \( g \) with \( f \).

This is written \( h = g \circ f \) (read ‘composition of \( f \) followed by \( g \)’) and the rule for \( h \) is defined by \( h(x) = g(f(x)) \). In the example we have considered:

\[
h(x) = g(f(x))
= g(3x + 2)
= (3x + 2)^2
\]

The domain of the function \( h = g \circ f = \text{domain of } f \).

In general for the composition of \( g \) with \( f \) to be defined, range of \( f \subseteq \text{domain of } g \).

When this composition (or composite function) of \( g \) with \( f \) is defined it is denoted \( g \circ f \).

For functions \( f \) and \( g \) with domains \( X \) and \( Y \) respectively and such that the range of \( f \subseteq Y \), we define the composite function of \( g \) with \( f \):

\[
g \circ f : X \rightarrow R, \text{ where } g \circ f(x) = g(f(x))
\]
Example 14

Find both \( f \circ g \) and \( g \circ f \), stating the domain and range of each where:

\[
f: R \to R, \quad f(x) = 2x - 1 \quad \text{and} \quad g: R \to R, \quad g(x) = 3x^2
\]

Solution

To determine the existence of a composite function, it is useful to form a table of domains and ranges.

<table>
<thead>
<tr>
<th></th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g )</td>
<td>( R )</td>
<td>( R^+ \cup {0} )</td>
</tr>
<tr>
<td>( f )</td>
<td>( R )</td>
<td>( R )</td>
</tr>
</tbody>
</table>

\( f \circ g \) is defined since \( \text{ran } g \subseteq \text{dom } f \), and \( g \circ f \) is defined since \( \text{ran } f \subseteq \text{dom } g \).

\[
f \circ g(x) = f(g(x)) = f(3x^2) = 2(3x^2) - 1 = 6x^2 - 1
\]

and \( \text{dom } f \circ g = \text{dom } g = R \) and \( \text{ran } f \circ g = [-1, \infty) \)

\[
g \circ f(x) = g(f(x)) = g(2x - 1) = 3(2x - 1)^2 = 12x^2 - 12x + 3
\]

\( \text{dom } g \circ f = \text{dom } f = R \)

\( \text{ran } g \circ f = [0, \infty) \)

It can be seen from this example that in general \( f \circ g \neq g \circ f \).

Using the TI-Nspire

Define \( f(x) = 2x - 1 \) and \( g(x) = 3x^2 \). The rules for \( f \circ g \) and \( g \circ f \) can now be found using \( f(g(x)) \) and \( g(f(x)) \).
Define \( f(x) = 2x - 1 \) and \( g(x) = 3x^2 \).
The rules for \( f \circ g \) and \( g \circ f \) can now be found using \( f(g(x)) \) and \( g(f(x)) \).

**Example 15**

For the functions \( g(x) = 2x - 1, x \in \mathbb{R} \) and \( f(x) = \sqrt{x}, x \geq 0 \):

a) State which of \( f \circ g \) and \( g \circ f \) is defined.
b) For the composite function that is defined, state the domain and rule.

**Solution**

a) Range of \( f \subseteq \) domain of \( g \)
   but range of \( g \nsubseteq \) domain of \( f \).
   \( \therefore g \circ f \) is defined but \( f \circ g \) is not defined.

\[
\begin{array}{c|c|c}
\text{Domain} & \text{Range} \\
g & \mathbb{R} & \mathbb{R} \\
f & \mathbb{R}^+ \cup \{0\} & \mathbb{R}^+ \cup \{0\}
\end{array}
\]

b) \( g \circ f(x) = g(f(x)) \)
   \[= g(\sqrt{x}) \]
   \[= 2\sqrt{x} - 1 \]
   \( \text{dom } g \circ f = \text{dom } f = \mathbb{R}^+ \cup \{0\} \)

**Example 16**

For the functions \( f(x) = x^2 - 1, x \in \mathbb{R} \), and \( g(x) = \sqrt{x}, x \geq 0 \):

a) State why \( g \circ f \) is not defined.
b) Define a restriction \( f^* \) of \( f \) such that \( g \circ f^* \) is defined and find \( g \circ f^* \).
Solution

a  Range of \( f \not\subseteq \) domain of \( g \).
\[
\therefore \quad g \circ f \text{ is not defined.}
\]

\[
\begin{array}{|c|c|}
\hline
\text{Domain} & \text{Range} \\
\hline
f & R \\
g & R^+ \cup \{0\} \\
\hline
\end{array}
\]

b  For \( g \circ f \) to be defined, range of \( f \subseteq \) domain of \( g \), i.e. range of \( f \subseteq R^+ \cup \{0\} \). For range of \( f \) to be a subset of \( R^+ \cup \{0\} \), the domain of \( f \)
must be restricted to a subset of:
\[
\{x: x \leq -1\} \cup \{x: x \geq 1\}, \text{ or } R \backslash (-1, 1).
\]

So we define \( f^* \) by:
\[
\begin{align*}
f^*: R \backslash (-1, 1) & \to R, f^*(x) = x^2 - 1 \\
g \circ f^*(x) &= g(f^*(x)) \\
&= g(x^2 - 1) \\
&= \sqrt{x^2 - 1}
\end{align*}
\]

\[
\text{dom } g \circ f^* = \text{dom } f^* = R \backslash (-1, 1)
\]

The composite function \( g \circ f^* \) is:
\[
g \circ f^*: R \backslash (-1, 1) \to R, g \circ f^*(x) = \sqrt{x^2 - 1}
\]

Compositions involving the modulus function

Functions with rules of the form \( y = |f(x)| \) and \( y = f(|x|) \) are considered in this section.

Functions of the form \( y = |f(x)| \)

\( |f| \) is the composition \( g \circ f \) where \( g(x) = |x| \). The function \( f \) is applied first and then the modulus function. The following observation enables the graph of functions with rule of the form \( y = |f(x)| \) to be sketched if the graph of \( y = f(x) \) is known:

\[
|f(x)| = f(x) \text{ if } f(x) \geq 0 \quad \text{ and } \quad |f(x)| = -f(x) \text{ if } f(x) < 0
\]

Example 17

Sketch the graphs of each of the following:
\[
a \quad y = |x^2 - 4| \quad \quad b \quad y = |2^x - 1|
\]
Chapter 1 — Functions and relations

Solution

a The graph of \( y = x^2 - 4 \) is drawn and the negative part reflected in the \( x \)-axis.

b The graph of \( y = 2^x - 1 \) is drawn and the negative part reflected in the \( x \)-axis.

Functions of the form \( y = f(|x|) \)

The graphs of functions with rules of the form \( y = f(|x|) \) where \( x \in \mathbb{R} \) are sketched by reflecting the graph of \( y = f(x) \), for \( x \geq 0 \), in the \( y \)-axis. The function with rule \( f(|x|) \) is the result of the composition \( f \circ g \) where \( g(x) = |x| \).

Example 18

Sketch the graphs of each of the following:

a \( y = |x|^2 - 2|x| \) (This is the rule for the function \( f \circ g \) where \( f(x) = x^2 - 2x \) and \( g(x) = |x| \).)

b \( y = 2^{|x|} \) (This is the rule for the function \( f \circ g \) where \( f(x) = 2^x \) and \( g(x) = |x| \).)

Solution

a The graph of \( y = x^2 - 2x, x \geq 0 \), is reflected in the \( y \)-axis.

b The graph of \( y = 2^x, x \geq 0 \), is reflected in the \( y \)-axis.
For each of the following, find \( f(g(x)) \) and \( g(f(x)) \):

1. \( f(x) = 2x - 1, \ g(x) = 2x \) 
2. \( f(x) = 4x + 1, \ g(x) = 2x + 1 \) 
3. \( f(x) = 2x - 1, \ g(x) = 2x - 3 \) 
4. \( f(x) = 2x^2 + 1, \ g(x) = x - 5 \) 
5. \( f(x) = 2x - 1, \ g(x) = x^2 \) 
6. \( f(x) = 2x + 1, \ g(x) = |x| \)

For the functions \( f(x) = 2x - 1 \) and \( h(x) = 3x + 2 \), find:

7. \( f \circ h(x) \) 
8. \( h(f(x)) \) 
9. \( f \circ h(2) \) 
10. \( h \circ f(2) \)

For the functions \( f(x) = x^2 + 2x \) and \( h(x) = 3x + 1 \), find:

11. \( f \circ h(x) \) 
12. \( h \circ f(x) \) 
13. \( f \circ h(3) \) 
14. \( h \circ f(3) \) 
15. \( f \circ h(0) \) 
16. \( h \circ f(0) \)

For the functions \( h: R \setminus \{0\} \to R, h(x) = \frac{1}{x^2} \) and \( g: R^+ \to R, g(x) = 3x + 2 \), find:

17. \( h \circ g \) (state rule and domain) 
18. \( g \circ h \) (state rule and domain) 
19. \( h \circ g(1) \) 
20. \( g \circ h(1) \)

For the functions \( f: R \to R, f(x) = x^2 - 4 \) and \( g: R^+ \cup \{0\} \to R, g(x) = \sqrt{x} \),

21. a) State the ranges of \( f \) and \( g \). 
22. b) Find \( f \circ g \), stating its range. 
23. c) Explain why \( g \circ f \) does not exist.

Let \( f \) and \( g \) be functions given by:

\[
\begin{align*}
 f: R \setminus \{0\} \to R, & \quad f(x) = \frac{1}{2} \left( \frac{1}{x} + 1 \right) \\
 g: R \setminus \left\{ \frac{1}{2} \right\} \to R, & \quad g(x) = \frac{1}{2x - 1}
\end{align*}
\]

Find:

24. a) \( f \circ g \) 
25. b) \( g \circ f \), and state the range in each case.

The functions \( f \) and \( g \) are defined by \( f: R \to R, f(x) = x^2 - 2 \) and \( g: \{x: x \geq 0\} \to R, \) where \( g(x) = \sqrt{x} \).

26. a) Explain why \( g \circ f \) does not exist. 
27. b) Find \( f \circ g \) and sketch its graph.

8. a) For \( f(x) = 4 - x \) and \( g(x) = |x| \), find \( f \circ g \) and \( g \circ f \) and sketch the graphs of each of these functions.
8. b) For \( f(x) = 9 - x^2 \) and \( g(x) = |x| \), find \( f \circ g \) and \( g \circ f \) and sketch the graphs of each of these functions.
8. c) For \( f: R \setminus \{0\} \to R, f(x) = \frac{1}{x} \) and \( g: R \setminus \{0\} \to R, g(x) = |x| \), find \( f \circ g \) and \( g \circ f \) and sketch the graphs of each of these functions.
9 \( f: \{ x: x \leq 3 \} \to R, \quad f(x) = 3 - x \) and \( g: R \to R, \quad g(x) = x^2 - 1 \)
   a Show that \( f \circ g \) is not defined.
   b Define a restriction \( g^* \) of \( g \) such that \( f \circ g^* \) is defined and find \( f \circ g^* \).

10 \( f: R^+ \to R, \quad f(x) = \frac{1}{2} \) and \( g: R \to R, \quad g(x) = 3 - x \)
   a Show that \( f \circ g \) is not defined.
   b By suitably restricting the domain of \( g \), obtain a function \( g_1 \) such that \( f \circ g_1 \) is defined.

11 Let \( f: R \to R, \quad f(x) = x^2 \) and let \( g: \{ x: x \leq 3 \} \to R, \quad g(x) = \sqrt{3 - x} \).
   a State with reasons whether:
       a \( f \circ g \) exists
       b \( g \circ f \) exists
   b Find the range of \( f \) and the range of \( g \).
   c State whether or not \( f \circ g \) and \( g \circ f \) are defined and give a reason for each assertion.

12 Let \( f: S \to R, \quad f(x) = \sqrt{4 - x^2} \) and \( S \) be the set of all real values of \( x \) for which \( f(x) \) is defined. Let \( g: R \to R, \quad g(x) = x^2 + 1 \).
   a Find \( S \).
   b Find the range of \( f \) and the range of \( g \).
   c State whether or not \( f \circ g \) and \( g \circ f \) are defined and give a reason for each assertion.

13 Let \( a \) be a positive number, let \( f: [2, \infty) \to R, \quad f(x) = a - x \) and let \( g: (-\infty, 1] \to R, \quad g(x) = x^2 + a \).
   Find all values of \( a \) for which \( f \circ g \) and \( g \circ f \) both exist.

### 1.7 Inverse functions

If \( f \) is a one-to-one function, then for each number \( y \) in the range of \( f \) there is exactly one number, \( x \), in the domain of \( f \) such that \( f(x) = y \).

Thus if \( f \) is a one-to-one function, a new function \( f^{-1} \), called the inverse of \( f \), may be defined by:

\[
  f^{-1}(x) = y \text{ if } f(y) = x, \text{ for } x \in \text{ran } f, y \in \text{dom } f
\]

It is not difficult to see what the relation between \( f \) and \( f^{-1} \) means geometrically. The point \((x, y)\) is on the graph of \( f^{-1} \) if the point \((y, x)\) is on the graph of \( f \). Therefore, to get the graph of \( f^{-1} \) from the graph of \( f \), the graph of \( f \) is to be reflected in the line \( y = x \).

From this the following is evident:

\[
  \text{dom } f^{-1} = \text{ran } f
\]
\[
  \text{ran } f^{-1} = \text{dom } f
\]

A function has an inverse function if and only if it is one-to-one.

We note \( f \circ f^{-1}(x) = x \), for all \( x \in \text{dom } f^{-1} \)
\( f^{-1} \circ f(x) = x \), for all \( x \in \text{dom } f \)
Example 19

Find the inverse function \( f^{-1} \) of the function \( f(x) = 2x - 3 \).

Solution

Method 1
The graph of \( f \) has equation \( y = 2x - 3 \) and the graph of \( f^{-1} \) has equation \( x = 2y - 3 \), i.e. \( x \) and \( y \) are interchanged. Solve for \( y \).

\[
x + 3 = 2y
\]
and
\[
y = \frac{1}{2}(x + 3)
\]
\[
\therefore f^{-1}(x) = \frac{1}{2}(x + 3)
\]
and dom \( f^{-1} = \text{ran } f = \mathbb{R} \)

Method 2
We require \( f^{-1} \) such that:

\[
f(f^{-1}(x)) = x
\]
\[
\therefore 2f^{-1}(x) - 3 = x
\]
\[
\therefore f^{-1}(x) = \frac{1}{2}(x + 3)
\]
and dom \( f^{-1} = \text{ran } f = \mathbb{R} \)

Example 20

\( f \) is the function defined by \( f(x) = \frac{1}{x^2}, x \in \mathbb{R}\setminus\{0\} \). Define a suitable restriction for \( f, f^* \), such that \( f^*-1 \) exists.

Solution

\( f \) is not a one-to-one function. Therefore the inverse function \( f^{-1} \) is not defined. The following restricted functions of \( f \) are one-to-one.

\[
f_1: (0, \infty) \rightarrow \mathbb{R}, \quad f_1(x) = \frac{1}{x^2} \quad \text{Range } f_1 = (0, \infty)
\]

\[
f_2: (-\infty, 0) \rightarrow \mathbb{R}, \quad f_2(x) = \frac{1}{x^2} \quad \text{Range } f_2 = (0, \infty)
\]

Let \( f^* \) be \( f_1 \) and determine \( f_1^{-1} \).
Method 1
Interchanging $x$ and $y$:

$$x = \frac{1}{y^2}$$
$$y^2 = \frac{1}{x}$$

$$\therefore y = \pm \frac{1}{\sqrt{x}}$$

But range $f^{-1}_1 = \text{domain } f_1 = (0, \infty)$

$$\therefore f^{-1}_1 = \frac{1}{\sqrt{x}}, \text{ ran } f^{-1}_1 = (0, \infty) \text{ and dom } f^{-1}_1 = \text{ ran } f_1 = (0, \infty)$$

$$\therefore f^{-1}_1: (0, \infty) \rightarrow R, f^{-1}_1(x) = \frac{1}{\sqrt{x}}$$

Method 2
We require $f^{-1}_1$ such that:

$$f_1 \left( f^{-1}_1(x) \right) = x$$

$$\therefore \left[ f^{-1}_1(x) \right]^2 = x$$

$$\therefore f^{-1}_1(x) = \pm \frac{1}{\sqrt{x}}$$

But range $f^{-1}_1 = \text{domain } f_1 = (0, \infty)$

$$\therefore f^{-1}_1 = \frac{1}{\sqrt{x}}, \text{ ran } f^{-1}_1 = (0, \infty) \text{ and dom } f^{-1}_1 = \text{ ran } f_1 = (0, \infty)$$

$$\therefore f^{-1}_1: (0, \infty) \rightarrow R, f^{-1}_1(x) = \frac{1}{\sqrt{x}}$$

Exercise 1G

1 For each of the following, find the rule for the inverse:

a $f: R \rightarrow R, f(x) = x - 4$

b $f: R \rightarrow R, f(x) = 2x$

c $f: R \rightarrow R, f(x) = \frac{3x}{4}$

d $f: R \rightarrow R, f(x) = \frac{3x - 2}{4}$

2 Find the inverse of each of the following functions, stating the domain and range for each:

a $f: [-2, 6] \rightarrow R, f(x) = 2x - 4$

b $g(x) = \frac{1}{9 - x}, x > 9$

c $h(x) = x^2 + 2, x \geq 0$

d $f: [-3, 6] \rightarrow R, f(x) = 5x - 2$

e $g: (1, \infty) \rightarrow R, g(x) = x^2 - 1$

f $h: R^+ \rightarrow R, h(x) = \sqrt{x}$
3. Consider the function \( g: [-1, \infty) \rightarrow \mathbb{R} \), \( g(x) = x^2 + 2x \).
   a. Find \( g^{-1} \), stating the domain and range.
   b. Sketch the graph of \( g^{-1} \).

4. Let \( f: S \rightarrow \mathbb{R} \), where \( S = \{x: 0 \leq x \leq 3\} \) and \( f(x) = 3 - 2x \). Find \( f^{-1}(2) \) and the domain of \( f^{-1} \).

5. Find the inverse of each of the following functions, stating the domain and range of each:
   a. \( f: [-1, 3] \rightarrow \mathbb{R} \), \( f(x) = 2x \)
   b. \( f: [0, \infty) \rightarrow \mathbb{R} \), \( f(x) = 2x^2 - 4 \)
   c. \( \{(1, 6), (2, 4), (3, 8), (5, 11)\} \)
   d. \( h: R^- \rightarrow \mathbb{R} \), \( h(x) = \sqrt{-x} \)
   e. \( f: R \rightarrow \mathbb{R} \), \( f(x) = x^3 + 1 \)
   f. \( g: (-1, 3) \rightarrow \mathbb{R} \), \( g(x) = (x + 1)^2 \)
   g. \( g: [1, \infty) \rightarrow \mathbb{R} \), \( g(x) = \sqrt{x - 1} \)
   h. \( h: [0, 2] \rightarrow \mathbb{R} \), \( h(x) = \sqrt{4 - x^2} \)

6. For each of the following functions, sketch the graph of the function and on the same set of axes sketch the graph of the inverse function. For each of the functions state the rule, domain and range of the inverse. It is advisable to draw in the line with equation \( y = x \) for each set of axes.
   a. \( y = 2x + 4 \)
   b. \( f(x) = \frac{3 - x}{2} \)
   c. \( f: [2, \infty) \rightarrow \mathbb{R} \), \( f(x) = (x - 2)^2 \)
   d. \( f: [1, \infty) \rightarrow \mathbb{R} \), \( f(x) = (x - 1)^2 \)
   e. \( f: (-\infty, 2] \rightarrow \mathbb{R} \), \( f(x) = (x - 2)^2 \)
   f. \( f: R^+ \rightarrow \mathbb{R} \), \( f(x) = \frac{1}{x} \)
   g. \( f: R^+ \rightarrow \mathbb{R} \), \( f(x) = \frac{1}{x^2} \)
   h. \( h(x) = \frac{1}{2(x - 4)} \)

7. Copy each of the following graphs and on the same set of axes draw the inverse of each of the corresponding functions:
8 Match each of the graphs of a, b, c and d with its inverse.

a

b

c

d

A

B

C

D
Let \( f : A \rightarrow R \), \( f(x) = \sqrt{3 - x} \). If \( A \) is the set of all real values of \( x \) for which \( f(x) \) is defined, find \( A \).

Let \( g : [b, 2] \rightarrow R \) where \( g(x) = 1 - x^2 \). If \( b \) is the smallest real value such that \( g \) has a
inverse function, find \( b \) and \( g^{-1}(x) \).

### 1.8 Applications

#### Example 21

The cost of a taxi trip in a particular city is $1.75 up to and including 1 km. After 1 km the passenger pays an additional 75 cents per kilometre. Find the function \( f \) which describes this method of payment and sketch the graph of \( y = f(x) \).

**Solution**

Let \( x \) denote the length of the trip.

Then \( f(x) = \begin{cases} 1.75 \text{ for } 0 \leq x \leq 1 \\ 1.75 + 0.75(x - 1) \text{ for } x > 1 \end{cases} \)

![Graph of y = f(x)](image)

#### Example 22

A rectangular piece of cardboard has dimensions 18 cm by 24 cm. Four squares each \( x \) cm by \( x \) cm are cut from the corners. An open box is formed by folding up the flaps.

Find a function for \( V \), which gives the volume of the box in terms of \( x \), and state the domain of the function.

**Solution**

The dimensions of the box will be:

\[ 24 - 2x, 18 - 2x \text{ and } x \]

The volume of the box is determined by the function:

\[ V(x) = (24 - 2x)(18 - 2x)x \]
For the box to be formed:

\[ 24 - 2x > 0 \quad \text{and} \quad 18 - 2x > 0 \quad \text{and} \quad x > 0 \]

Therefore, \( x < 12 \) and \( x < 9 \) and \( x > 0 \). Therefore the domain is \((0, 9)\).

**Example 23**

A rectangle is inscribed in an isosceles triangle with the dimensions as shown.

Find an area of the rectangle function and state the domain.

**Solution**

Let the height of the rectangle be \( y \) cm and the width \( 2x \) cm.

The height (\( h \) cm) of the triangle can be determined by Pythagoras’ theorem:

\[ h = \sqrt{15^2 - 9^2} = 12 \]

In the diagram to the right, \( AYX \sim ABD \) (i.e., triangle \( AYX \) is similar to triangle \( ABD \)).

Therefore \( \frac{x}{9} = \frac{12 - y}{12} \)

and \( \frac{12x}{9} = 12 - y \)

which implies \( y = 12 - \frac{12x}{9} \)

The area of the rectangle, \( A = 2xy \)

and \( A(x) = 2x \left( 12 - \frac{12x}{9} \right) \)

For the rectangle to be formed,

\( x > 0 \) and \( 12 - \frac{12x}{9} > 0 \)

\( \therefore x > 0 \) and \( x < 9 \)

The domain of the function is \((0, 9)\).

\[ A: (0, 9) \rightarrow R, \quad A(x) = 2x \left( 12 - \frac{12x}{9} \right) = \frac{24x}{9}(9 - x) \]
1 The cost of a taxi trip in a particular city is $4.00 up to and including 2 km. After 2 km the passenger pays an additional $2.00 per kilometre. Find the function \( f \) which describes this method of payment and sketch the graph of \( y = f(x) \), where \( x \) is the number of kilometres travelled. (Use a continuous model.)

2 A rectangular piece of cardboard has dimensions 20 cm by 36 cm. Four squares each \( x \) cm by \( x \) cm are cut from the corners. An open box is formed by folding up the flaps. Find a function for \( V \), which gives the volume of the box in terms of \( x \), and state the domain for the function.

3 The dimensions of an enclosure are shown. The perimeter of the enclosure is 160 m.
   a Find a rule for the area, \( A \) m\(^2\), of the enclosure in terms of \( x \).
   b State a suitable domain of the function \( A(x) \).
   c Sketch the graph of \( A \) against \( x \).
   d Find the maximum possible area of the enclosure and state the corresponding values of \( x \) and \( y \).

4 A cuboid tank is open at the top and the internal dimensions of its base are \( x \) m and \( 2x \) m. The height is \( h \) m. The volume of the tank is \( V \) cubic metres and the volume is fixed. Let \( S \) m\(^2\) denote the internal surface area of the tank.
   a Find \( S \) in terms of:
      i \( x \) and \( h \)
      ii \( V \) and \( x \)
   b State the maximal domain for the function defined by the rule in a ii.
   c If \( 2 < x < 15 \) find the maximum value of \( S \) if \( V = 1000 \) m\(^3\).

5 A man walks at a speed of 2 km/h for 45 minutes and then runs at 4 km/h for 30 minutes. Let \( S \) km be the distance the man has run after \( t \) minutes. The distance travelled can be described by:
   \[
   S(t) = \begin{cases} 
   at & \text{if } 0 \leq t \leq c \\
   bt + d & \text{if } c < t \leq e 
   \end{cases}
   \]
   a Find the values \( a, b, c, d, e \).
   b Sketch the graph of \( S(t) \) against \( t \).
   c State the range of the function.

6 Suppose Australia Post charged the following rates for airmail letters to Africa: $1.20 up to 20 g; $2.00 over 20 g and up to 50 g; $3.00 over 50 g and up to 150 g.
   a Write a cost function, \( C(\)\$\), in terms of the mass, \( m \) (g) for letters up to 150 g.
   b Sketch the graph of the function, stating the domain and range.
Chapter summary

A relation is a set of ordered pairs.

The domain of a relation $S$ is the set of all first elements of the ordered pairs in $S$. The domain of $f$ is denoted by $\text{dom } f$.

The range of a relation $S$ is the set of all the second elements of the ordered pairs in $S$. The range of $f$ is denoted by $\text{ran } f$.

A function is a relation such that no two ordered pairs of the relation have the same first element.

For each $x$ in the domain of a function $f$ there is a unique element, $y$, in the range such that $(x, y) \in f$. The element $y$ is called the image of $x$ under $f$ or the value of $f$ at $x$ and is denoted by $f(x)$.

A function $f$ is said to be one-to-one if for $a, b \in \text{dom } f, a \neq b$, then $f(a) \neq f(b)$.

A function which is not one-to-one is many-to-one.

If the domain of a function is unspecified, then the domain is the largest subset of $R$ for which the rule is defined. This set is called the implied domain or maximal domain of the function rule.

A function $f$ is odd if $f(-x) = -f(x)$ for all $x$ in the domain of $f$.

A function $f$ is even if $f(-x) = f(x)$ for all $x$ in the domain of $f$.

Let $f$ and $g$ be functions such that $\text{dom } f \cap \text{dom } g \neq \emptyset$. The sum $f + g$, and the product, $fg$, as functions on $\text{dom } f \cap \text{dom } g$ are defined by:

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (fg)(x) = f(x) \cdot g(x)$$

If $f$ is a one-to-one function, a new function, $f^{-1}$, called the inverse of $f$, may be defined by:

$$f^{-1}(x) = y \text{ if } f(y) = x, \text{ for } x \in \text{ran } f, y \in \text{dom } f$$

For a function $f$ and its inverse $f^{-1}$:

$$\text{dom } f^{-1} = \text{ran } f$$
$$\text{ran } f^{-1} = \text{dom } f$$

The inverse of any relation may be defined. The inverse relation is not a function unless the initial relation is a one-to-one function.

For a relation $S = \{(a, b)\}$ the inverse relation is $\{(b, a)\}$.

The modulus or absolute value of a real number $x$ is denoted by $|x|$ and is defined by:

$$|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases}$$

It may also be defined as $|x| = \sqrt{x^2}$.
The function $|x|$ has the following properties:

- $|ab| = |a||b|$
- $\frac{a}{b} = \frac{|a|}{|b|}$
- $|a + b| \leq |a| + |b|$. If $a$ and $b$ are both non-negative or both non-positive, then equality holds.
- If $a \geq 0$, $|x| \leq a$ is equivalent to $-a \leq x \leq a$
- If $a \geq 0$, $|x - k| \leq a$ is equivalent to $k - a \leq x \leq k + a$
- The composition of functions $f$ and $g$ is denoted by $f \circ g$. The rule is given by $f(g(x))$. The domain of $f \circ g$ is the domain of $g$. The composition $f \circ g$ is defined if the range of $g \subseteq$ the domain of $f$.

### Multiple-choice questions

1. For the function with rule $f(x) = \sqrt{6 - 2x}$, which of the following is the maximal domain?
   - A $(-\infty, 6]$  
   - B $[3, \infty)$  
   - C $(-\infty, 6]$  
   - D $(3, \infty)$  
   - E $(-\infty, 3]$  

2. For $f : [-1, 3) \to \mathbb{R}$, $f(x) = -x^2$, the range is:
   - A $R$  
   - B $(-9, 0]$  
   - C $(-\infty, 0]$  
   - D $(-9, 1]$  
   - E $[-9, 0]$  

3. For $f(x) = 3x^2 + 2x$, $f(2a) =$
   - A $20a^2 + 4a$  
   - B $6a^2 + 2a$  
   - C $6a^2 + 4a$  
   - D $36a^2 + 4a$  
   - E $12a^2 + 4a$  

4. For $f(x) = 2x - 3$, $f^{-1}(x) =$
   - A $2x + 3$  
   - B $\frac{1}{2}x + 3$  
   - C $\frac{1}{2}x + \frac{3}{2}$  
   - D $\frac{1}{2x - 3}$  
   - E $\frac{1}{2}x - 3$  

5. For $f : (a, b) \to \mathbb{R}$, $f(x) = 10 - x$ where $a < b$ the range is:
   - A $(10 - a, 10 - b)$  
   - B $(10 - a, 10 - b]$  
   - C $(10 - b, 10 - a)$  
   - D $(10 - b, 10 - a]$  
   - E $[10 - b, 10 - a)$  

6. For the function with rule
   
   $f(x) = \begin{cases} 
   x^2 + 5 & \text{if } x \geq 3 \\
   -x + 6 & \text{if } x < 3
   \end{cases}$

   the value of $f(a + 3)$, where $a$ is a negative real number, is:
   - A $a^2 + 6a + 14$  
   - B $-a + 9$  
   - C $-a + 3$  
   - D $a^2 + 14$  
   - E $a^2 + 8a + 8$  

7. Which one of the following sets is a possible domain for the function with rule $f(x) = (x + 3)^2 - 6$ if the inverse function is to exist?
   - A $R$  
   - B $[-6, \infty)$  
   - C $(-\infty, 3]$  
   - D $[6, \infty)$  
   - E $(-\infty, 0]$  

8. If $f(x) = 3x^2$ and $g(x) = 2x + 1$, then $f(g(a))$ is equal to:
   - A $12a^2 + 3$  
   - B $12a^2 + 12a + 3$  
   - C $6a^2 + 1$  
   - D $6a^2 + 4$  
   - E $4a^2 + 4a + 1$
9 For which one of the following functions does an inverse function not exist?
   A $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x - 4$
   B $g: [-4, 4] \rightarrow \mathbb{R}, g(x) = \sqrt{16 - x^2}$
   C $h: [0, \infty) \rightarrow \mathbb{R}, h(x) = -\frac{1}{2}x^2$
   D $p: \mathbb{R}^+ \rightarrow \mathbb{R}, p(x) = \frac{1}{x^2}$
   E $q: \mathbb{R} \rightarrow \mathbb{R}, q(x) = 2x^3 - 5$

10 The graph of the function $f$ with rule $y = f(x)$ is shown on the right.

Which one of the following is most likely to be the graph of the inverse function of $f$?

A

B

C

D

E

Short-answer questions (technology-free)

1 Sketch the graph of each of the following relations and state the implied domain and range:
   a $f(x) = x^2 + 1$
   b $f(x) = 2x - 6$
   c $\{(x, y): x^2 + y^2 = 25\}$
   d $\{(x, y): y \geq 2x + 1\}$
   e $\{(x, y): y < x - 3\}$
2 For the function \( g: [0, 5] \to R, g(x) = \frac{x + 3}{2} \):
   a Sketch the graph of \( y = g(x) \).
   b State the range of \( g \).
   c Find \( g^{-1} \), stating the domain and range of \( g^{-1} \).
   d Find \( \{x: g(x) = 4\} \).
   e Find \( \{x: g^{-1}(x) = 4\} \).

3 For \( g(x) = 5x + 1, \) find:
   a \( \{x: g(x) = 2\} \)
   b \( \{x: g^{-1}(x) = 2\} \)
   c \( \left\{ x: \frac{1}{g(x)} = 2 \right\} \)

4 Sketch the graph of the function \( f \), for which
   \[
   f(x) = \begin{cases} 
   x + 1 & \text{for } x > 2 \\
   x^2 - 1 & \text{for } 0 \leq x \leq 2 \\
   -x^2 & \text{for } x < 0 
   \end{cases}
   \]

5 Find the implied domains for each of the following:
   a \( f(x) = \frac{1}{2x - 6} \)
   b \( g(x) = \frac{1}{\sqrt{x^2 - 5}} \)
   c \( h(x) = \frac{1}{(x - 1)(x + 2)} \)
   d \( h(x) = \sqrt{25 - x^2} \)
   e \( f(x) = \sqrt{x - 5} + \sqrt{15 - x} \)
   f \( h(x) = \frac{1}{3x - 6} \)

6 For \( f(x) = (x + 2)^2 \) and \( g(x) = x - 3 \), find \((f + g)(x)\) and \((fg)(x)\).

7 For \( f: [1, 5] \to R, f(x) = (x - 1)^2, g: R \to R, g(x) = 2x \), find \( f + g \) and \( fg \).

8 For \( f: [3, \infty) \to R, f(x) = x^2 - 1 \), find \( f^{-1} \).

9 For \( f(x) = 2x + 3 \) and \( g(x) = -x^2 \) find:
   a \( (f + g)(x) \)
   b \( (fg)(x) \)
   c \( \{x: (f + g)(x) = 0\} \)

10 Let \( f: (-\infty, 2] \to R, f(x) = 3x - 4 \). On the one set of axes, sketch the graph of \( y = f(x) \) and \( y = f^{-1}(x) \).

### Extended-response questions

1 Self-Travel, a car rental firm, has two methods of charging for car rental:
   i method 1: $64 per day + 25 cents per kilometre
   ii method 2: $89 per day with unlimited travel.
   a Write a rule for each method if \( x \) kilometres per day are travelled and the cost in dollars is \( C_1 \) using method 1 and \( C_2 \) using method 2.
   b Draw the graph of each, using the same axes.
   c Determine, from the graph, the distance that must be travelled per day if method 2 is cheaper than method 1.

2 Express the total surface area, \( S \), of a cube as a function of:
   a the length \( l \) of an edge
   b the volume \( V \) of the cube

3 Express the area of an equilateral triangle as a function of:
   a the length \( s \) of each side
   b the altitude \( h \)
4 The base of a 3 m ladder leaning against a wall is \( x \) metres from the wall.
   a Express the distance, \( d \), from the top of the ladder to the ground as a function of \( x \) and sketch the graph of the function.
   b State the domain and range of the function.

5 A car travels half the distance of a journey at an average speed of 80 km/h and half at an average speed of \( x \) km/h.
Define a function, \( S \), which gives the average speed for the total journey as a function of \( x \).

6 A cylinder is inscribed in a sphere with a radius of length 6 cm.
For the cylinder:
   a Define a function, \( V_1 \), which gives the volume of the cylinder as a function of the height (\( h \)).
   (State the rule and domain.)
   b Define a function, \( V_2 \), which gives the volume of the cylinder as a function of the radius of the cylinder (\( r \)).
   (State the rule and domain.)

7 Let \( f: \mathbb{R} \to \mathbb{R} \) and \( g: \mathbb{R} \to \mathbb{R} \), where \( f(x) = x + 1 \) and \( g(x) = 2 + x^3 \).
   a State why \( g \circ f \) exists and find \( g \circ f(x) \).
   b State why \( g \circ f \) is a function and find \( (g \circ f)^{-1}(10) \).

8 A function \( f \) is defined as follows:
   \[
   f(x) = \begin{cases} 
   x^2 - 4, & \text{for } x \in (-\infty, 2) \\
   x, & \text{for } x \in [2, \infty) 
   \end{cases}
   \]
   a Sketch the graph of \( f \).
   b Find the value of:
      i \( f(-1) \)
      ii \( f(3) \)
   c Given \( g: S \to \mathbb{R} \) where \( g(x) = f(x) \), find the largest set \( S \) so that the inverse of \( g \) exists and \(-1 \in S \).
   d If \( h(x) = 2x \), find \( f(h(x)) \) and \( h(f(x)) \).

9 Find the rule for the area, \( A(t) \), enclosed by the graph of the function:
   \[
   f(x) = \begin{cases} 
   3x, & \text{for } 0 \leq x \leq 1 \\
   3, & \text{for } x > 1
   \end{cases}
   \]
the \( x \)-axis, the \( y \)-axis and the vertical line \( x = t \) (\( t \geq 0 \)). State the domain and range of the function.
10 Let \( f: \mathbb{R} \setminus \{ -\frac{d}{c} \} \to \mathbb{R}, f(x) = \frac{ax + b}{cx + d} \)

a Find the inverse function \( f^{-1} \).

b Find the inverse function when:

i \( a = 3, b = 2, c = 3, d = 1 \)

ii \( a = 3, b = 2, c = 2, d = -3 \)

iii \( a = 1, b = -1, c = -1, d = -1 \)

iv \( a = -1, b = 1, c = 1, d = 1 \)

c Determine the possible values of \( a, b, c \) and \( d \) if \( f = f^{-1} \).

11 The radius of the incircle of the right-angled triangle \( ABC \) is \( r \) cm. Find:

a i \( YB \) in terms of \( r \)

ii \( ZB \) in terms of \( r \)

iii \( AZ \) in terms of \( r \) and \( x \)

iv \( CY \)

b Use the geometric results \( CY = CX \) and \( AX = AZ \) to find an expression for \( r \) in terms of \( x \).

c i Find \( r \) when \( x = 4 \).

ii Find \( x \) when \( r = 0.5 \).

d Use a CAS calculator to investigate the possible values \( r \) can take.

12 Let \( f(x) = \frac{px + q}{x + r} \) where \( x \in \mathbb{R} \setminus \{-r, r\} \).

a If \( f(x) = f(-x) \) for all \( x \), show that \( f(x) = p \) for \( x \in \mathbb{R} \setminus \{-r, r\} \).

b If \( f(-x) = -f(x) \) for \( x \neq 0 \), find the rule for \( f(x) \) in terms of \( q \).

c If \( p = 3, q = 8 \) and \( r = -3 \):

i Find the inverse function of \( f \).

ii Find the values of \( x \) for which \( f(x) = x \).

13 a Let \( f(x) = \frac{x + 1}{x - 1} \).

i Find \( f(2), f(f(2)), f(f(f(2))) \).

ii Find \( f(f(x)) \).

b Let \( f(x) = \frac{x - 3}{x + 1} \).

Find \( f(f(x)), f(f(f(x))) \).